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Advanced Control

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3.1	29/10/2020	US	Final version

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a_k	coefficients for tyre formula, k = 1,2…5
a_x, a_y	longitudinal and lateral acceleration of the vehicle
$a_{xT,XY,CGT}, a_{yT,XY,CGT}$	longitudinal and lateral accelerations of trailer in X, Y
$a_{xT,CGT}, a_{yT,CGT}$	longitudinal and lateral accelerations of trailer in x_T, y_T
A	rotation matrix from inertial frame to x, y
BCD	stiffness at zero slip in Pacejka’s magic formula
B_{ij}	Pacejka’s magic formula stiffness factor of each wheel
C	Pacejka’s magic formula shape factor
C_x	aerodynamic drag coefficient
c	distance between hinge and centre of mass of the vehicle
CM	centre of mass
CG	centre of gravity vehicle
CGT	centre of gravity of the trailer
D_{ij}	Pacejka’s magic formula peak value of each wheel
d_F	distance between right and left wheel on the front
d_R	distance between right and left wheel on the rear
d_T	distance between right and left wheel on the trailer
eps	small smoothing parameter
$F_{x,ij}$	longitudinal force
$F_{y,ij}$	lateral force
$F_{x,j,T}, F_{y,j,T}$	longitudinal and lateral forces of the trailer
F_{TW}	friction resistance
$F_{z,ij,static}$	static component of vertical force on each wheel

$F_{z,ij,longit.}$	longitudinal component of vertical force on each wheel
F_z	normal force
f	rolling resistance coefficient
g	gravitational acceleration
h	height of centre of mass from the road surface
h_{HITCH}	height of the hitch joint from the road surface
H_{roll}	vertical distance between centre of mass and centre of roll of the vehicle
$H_{roll,T}$	vertical distance between centre of mass and centre of roll of the trailer
i, j, k	generic indexes, i = F (front), R (rear); j = L (left), R (right); k=T element with this superscript is referring to trailer
J_{KPI}	KPI cost function
J_{KPI}^{*}	KPI cost function optimal value
J_T	moment of inertia of the trailer
I_w	wheel moment of inertia of each wheel about its axis of rotation
J_z	moment of inertia of the vehicle
J	cost function
J_p	hitch joint point
$k_{roll,i}$	roll stiffness for the front and the rear of the vehicle
L_F	longitudinal distance of the front axle from centre of mass of the vehicle
L_R	longitudinal distance of the rear axle from centre of mass of the vehicle
$L_{F,T}$	longitudinal distance of the hinge axle from centre of mass of the trailer
$L_{R,T}$	longitudinal distance of the rear axle of the trailer from its centre of mass

L_{TOT}	distance between front axle and rear axle
$L_{TOT,T}$	distance between hinge and rear axle of the trailer
MF	Pacejka's Magic Formula
M	mass of the vehicle and trailer
M_z	yaw moment
$M_{z,ref}$	reference yaw moment
$M_{\Delta\dot{\psi}}$	normalisation factor $RMSE_{\Delta\dot{\psi}}$
$M_{\Delta\theta^*}$	normalisation factor $RMSE_{\Delta\theta^*}$
M_{IACA}	normalisation factor $IACA$
$M_{\alpha_R^{max}}$	normalisation factor α_R^{max}
$M_{\theta^{max}}$	normalisation factor θ^{max}
m	mass of the vehicle
m_T	mass of the trailer
m_R	slope of straight section
$P_{loss,xslip,ij}$	tyre slip power loss for each tyre in longitudinal direction
$P_{loss,yslip,ij}$	tyre slip power loss for each tyre in lateral direction
P_{LB}, P_{UB}	lower and upper boundary of tuning parameter
P_{opt}	tuning parameter
$P_{loss,xslip}^{tot}$	total tyre slip power loss in longitudinal direction
$P_{loss,yslip}^{tot}$	total tyre slip power loss in lateral direction
${P^{TR}}_{loss,EM}$	motor power loss in traction
${P^{REG}}_{loss,EM}$	motor power loss in regeneration
P_{mot}^{in}	input power to the electric motor
$P_{loss,INV}^{TR}$	power loss inverter in traction

$P_{loss,INV}^{REG}$	power loss inverter in regeneration
$P_{loss,PWT}$	powertrain power loss
q	generalized coordinates
$\dot{q}$	time derivative of generalized coordinates
q_R	straight section intercept
Q_i	generalized forces
R	wheel radius
R_T	rotation matrix inertial frame to trailer frame
RCH	height of the roll centre from the road surface of the vehicle
$RCHT$	height of the roll centre from the road surface of the trailer
R_{ZT}	vertical reaction of hitch joint
R_{yVy}	lateral force contribution of the hitch joint on the towing vehicle
R_{yT}	lateral force exchanged between the trailer and the hitch joint
R_{zV}	vertical force contribution of the hitch joint on the towing vehicle
S	frontal area of the vehicle
s_α	slack variable rear sideslip angle
s_θ	slack variable hitch angle error
s_{ij}	total slip
$s_{x,ij}$	longitudinal slip
$s_{y,ij}$	lateral slip
U	controlled input vector
V	vehicle velocity at its centre of mass
V_{CGT}	velocity in the CGT
V_J	velocity in the hitch joint

$V_{pi,j}$	velocity of the centre $P_{i,j}$ of the contact area
V_x	vehicle longitudinal velocity
V_y	vehicle lateral velocity
$v_{w,ij}$	wheel ground contact point velocity
$v_{slip,x,ij}$	longitudinal slip velocity of each wheel
$v_{slip,y,ij}$	lateral slip velocity of each wheel
X, Y	absolute inertial coordinates reference system
X_{CG}, Y_{CG}	CG coordinates of the towing vehicle
X_{J_p}, Y_{J_p}	inertial coordinates of hitch joint
X_V	vehicle state vector
$X_{V,d}$	vehicle desirable state vector
w	vector of generalized velocities
W	parameters vector
W_{s_θ}	slack variable weight inside the cost function
x, y	vehicle coordinates reference system
x_T, y_T	trailer coordinates reference system
Γ	damping coefficient for rotations
α_{ij}	slip angle of each tyre
$\alpha_{j,T}$	slip angle of the trailer
β	vehicle sideslip angle at its centre of mass
δ	steering wheel angle of the vehicle
δ_d	driver input steering wheel angle
δ_L	virtual work of the articulated vehicle
δL_T	virtual work of the trailer

δL_V	virtual work of the towing vehicle
$\delta_{pi,jx}, \delta_{pi,jy}$	virtual displacement in reference frame of towing vehicle
$\delta x, \delta y$	virtual linear displacement
$\delta\psi, \delta\theta$	virtual angular displacement
$\delta_{xT_{Pj}}, \delta_{yT_{Pj}}$	virtual displacement in trailer reference system
ϕ	roll angle
$\Delta\phi$	roll angle variation
$\Delta\dot{\psi}_\theta$	modified hitch angle error
$\Delta\dot{\psi}$	yaw rate error
$\Delta\theta_c$	modified hitch angle error
$\Delta\theta_{act}$	actual hitch angle error
$\Delta\theta_{bound}$	limit value to calculate $RMSE_{\Delta\theta^*}$
$\Delta\theta^*$	hitch angle error inside $RMSE_{\Delta\theta^*}$
$\Delta\theta_{th}$	hitch angle threshold error
$\Delta F_{z,aero,ij,k}$	load transfer due to aerodynamic forces
$\Delta F_{z,a_y,ij,k}$	lateral load transfer on each axle caused by lateral acceleration
$\Delta M_{antiroll}$	anti-roll moment
θ	hitch angle
$\dot{\theta}$	hitch rate
$\ddot{\theta}$	hitch angular acceleration
θ_{des}	desired hitch angle value
$\eta^{TR/REG}$	motor efficiency in traction or regeneration
η^{INV}	inverter efficiency
ω_{ij}	angular velocity of the driven/braked wheel

$\dot{\omega}_{ij}$	angular acceleration of the driven/braked wheel
μ_{ij}	resultant tyre force coefficient for each wheel
$\mu_{x,ij}$	longitudinal tyre force coefficient for each wheel
$\mu_{y,ij}$	lateral tyre force coefficient for each wheel
ψ	vehicle yaw angle at its centre of mass
$\dot{\psi}$	yaw rate
$\ddot{\psi}$	yaw acceleration
$\dot{\psi}_d$	reference yaw rate look up table
ρ	air density
τ_{ij}	torque on individual wheel
τ_{tot}	total axle torque
$\tau_{tot,d}$	driver total axle torque demand
T	kinetic energy

Introduction

This report covers the activities of Task 2.7 of SYSWHEEL. In particular, the report presents and analyses the energy-efficiency of a torque-vectoring (TV) control system for an electric front-wheel-drive commercial vehicle with in-wheel powertrains, using a non-linear model predictive control (NMPC). Moreover, four NMPC formulations, using hitch angle measurement for the TV control of the rigid vehicle towing a trailer, to reduce the articulation angle oscillation in emergency conditions, are presented. The NMPC is flexible and configurable as it includes the slip control function, and the cost function can incorporate various terms, e.g. related to the sideslip angle, to the longitudinal and lateral tyre slip power losses.

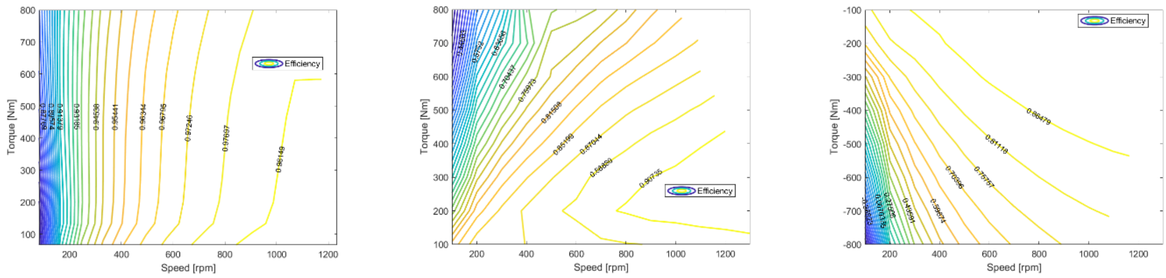
More specifically this report is organized as follows. Section 1 discusses the energy efficiency maps and the effect of direct yaw moment on powertrain power loss; section 2 describes the internal model of the rigid vehicle configuration and the optimal control problem formulation. Section 3 contains the mathematical derivation of the internal articulated vehicle model's equations and its optimal control problem definition. Moreover, the hitch angle control approaches including the respective tuning routine are presented. These novel formulations, due to the very promising results, will be included in a journal paper that is going to be submitted very shortly after this deliverable report.

1 Energy efficiency

The following section analyses whether the TV can be beneficial in terms of energy consumption showing efficiency maps and the effect of direct yaw moment on the powertrain power losses.

1.1 Efficiency maps

The figure below shows the efficiency maps for the inverter and the motor in traction and regeneration.



(a) **(b)** **(c)**

Figure 1. Experimentally measured (a) inverter efficiency, (b) motor efficiency in traction, (c) motor efficiency in regeneration.

The experimental data are provided from the project participant Elaphe. The components used are the motor M700 and the inverter H300.

1.2 Effect of direct yaw moment on powertrain power loss

In an EV with at least two EMs on the same axle, it is possible to generate either a destabilising yaw moment to reduce understeer, or a stabilising yaw moment to increase understeer. The direct yaw moment is provoked by the uneven torque distribution between the two sides of the EV. The torque demands on each side are calculated in the following equation from $\tau_{tot,d}$ and $M_{z,ref}$

$$\tau_{FR} = 0.5\tau_{tot,d} - M_{z,ref}\frac{R_w}{d_F} \quad (1)$$

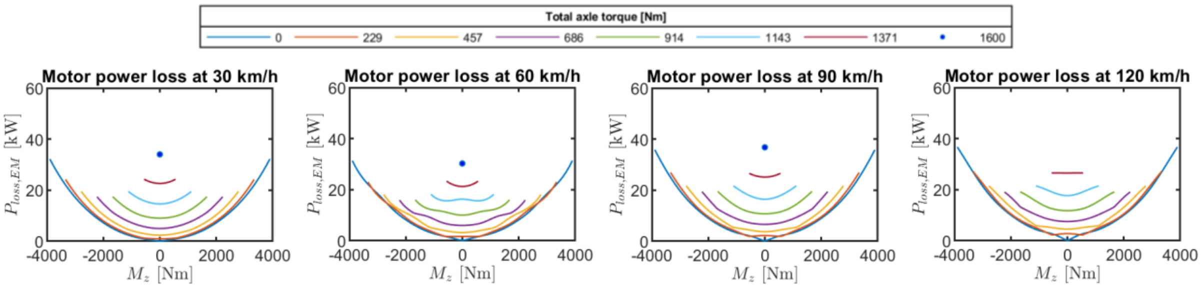
$$\tau_{FL} = 0.5\tau_{tot,d} + M_{z,ref}\frac{R_w}{d_F} \quad (2)$$

The total motor power loss can be expressed as sum of the power losses in traction and regeneration, where both these power losses are given by:

$$P_{loss,EM}^{TR} = \tau_{mot}\omega\left(\frac{1}{\eta^{TR}} - 1\right) \quad (3)$$

$$P_{loss,EM}^{REG} = \tau_{mot}\omega(1 - \eta^{REG}) \tag{4}$$

where τ_{mot} is the torque on each motor, ω is the angular speed of the wheel and η^{TR}, η^{REG} are the efficiency in traction and regeneration of the motors respectively. Here Figure 2 there are some examples of motor power loss with respect to the direct yaw moment.



(a) (b) (c) (d)

Figure 2. EM power loss as function of direct yaw moment for different total torque demand and vehicle speed of (a) 30 km/h, (b) 60 km/h, (c) 90 km/h and (d) 120 km/h.

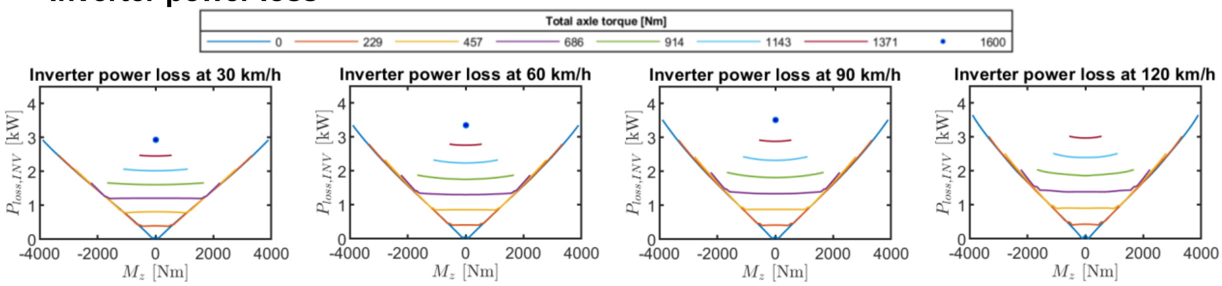
The direct yaw moment for these cases is calculated as follows:

$$M_z(\tau_{FL} = \tau_{tot,d}, \tau_{FR} = 0\,) = 0.5\tau_{tot,d}\frac{d_F}{R_w} \tag{5}$$

$$M_z\big(\tau_{FL} = 0, \tau_{FR} = \tau_{tot,d}\,\big) = -0.5\tau_{tot,d}\frac{d_F}{R_w} \tag{6}$$

From Figure 2 it is possible to see that for low speed values of the vehicle, the optimal solution is for zero yaw moment, whilst with the increasing of the speed it is possible to see that for low axle torque there are two local minima at yaw moment different from zero but for higher axle torque the minimum still remains at zero yaw moment.

- **Inverter power loss**



(a) (b) (c) (d)

Figure 3. Inverter power losses at different vehicle speed (a) 30 km/h, (b) 60 km/h, (c) 90 km/h, (d) 120 km/h.

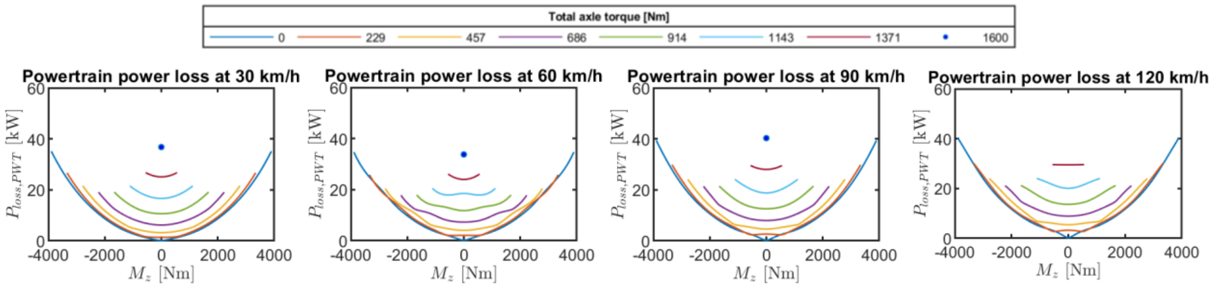
The inverter power losses are calculated as follows:

$$P_{loss,INV}^{TR} = P_{mot}^{in}\left(\frac{1}{\eta^{INV}} - 1\right) \quad (7)$$

$$P_{loss,INV}^{REG} = P_{mot}^{in}(1 - \eta^{INV}) \quad (8)$$

Where P_{mot}^{in} is the input power to the electric motor and η^{INV} is the efficiency of the inverter. From Figure 3 it is possible to see that the minimum power loss is achieved with zero yaw moment for low speed values, whilst with the increasing of the speed it is possible to see that for low axle torque there are two local minima at yaw moment different from zero but for higher axle torque the minimum still remains at zero yaw moment.

- **Powertrain power loss**



(a) (b) (c) (d)

Figure 4. Powertrain power losses at different vehicle speed (a) 30 km/h, (b) 60 km/h, (c) 90 km/h, (d) 120 km/h.

The powertrain power loss is given from the sum of the motor power loss and the inverter power loss. From Figure 4 it is possible to see that for low speed values of the vehicle, the optimal solution is for zero yaw moment, whilst with the increasing of the speed (60 km/h) it is possible to see that for low axle torque value, the minimum power loss is achieved with zero yaw moment but for higher axle torque (about 1100 Nm) there are two local minima at yaw moment different from zero but for higher torque the minimum still remains at zero yaw moment. Then at higher speed (up to 120 km/h) the optimal solution is still at zero yaw moment for almost all axle torque values, except for the low axle torque (about 220 Nm) where there are two minima at yaw moment different from zero.

1.3 Polynomial fitting – motor and inverter power losses

To calculate the power losses with the controller in the prediction horizon is not possible to use a lookup table, thus a novel formulation to compute the power losses was used. A polynomial of 7th grade for each torque value is created obtaining eight coefficients $(a_0, a_1, \dots, a_7)$ for each polynomial formula. After that, another 16th grade polynomial is created to approximate each coefficient's grade of the previous polynomials. Thus, the powertrain power loss is computed as follows:

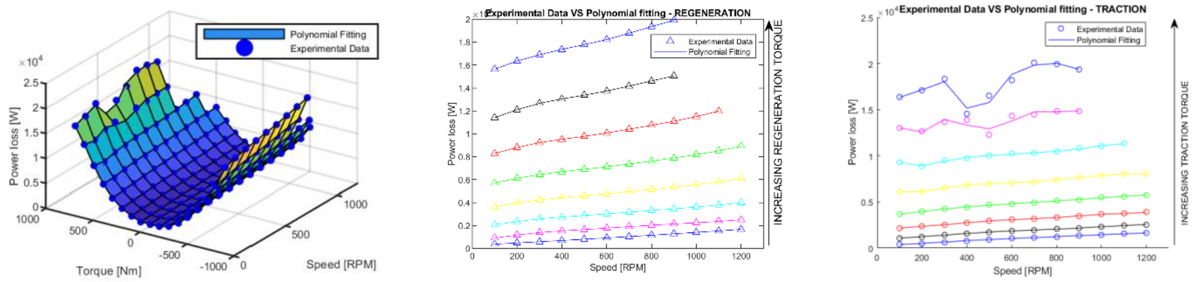
$$P_{loss,PWT}(\omega,\tau) = \sum_{k=0}^{n=16} (C_k \omega^k) \tag{9}$$

where C_k is expressed as follows:

$$C_k(\tau) = \sum_{l=0}^{m=7} (a_{kl} \tau^l) \tag{10}$$

where m and n are the grade of the polynomial, a_{kl} are the coefficients of the second polynomial which approximate the coefficients $(a_0, a_1, \dots, a_7)$ of each polynomial generated at each torque.

In this way the following results are obtained:



(a) (b) (c)

Figure 5. Polynomial fitting powertrain power loss, (a) 3D view, (b) 2D regeneration, (c) 2D traction.

From Figure 5 it is possible to see that now the powertrain power loss approximation is very close to the experimental data.

- **Relationship between torque demand and direct yaw moment**

Figure 6 indicates the maximum yaw moment usable with the actual vehicle configuration.

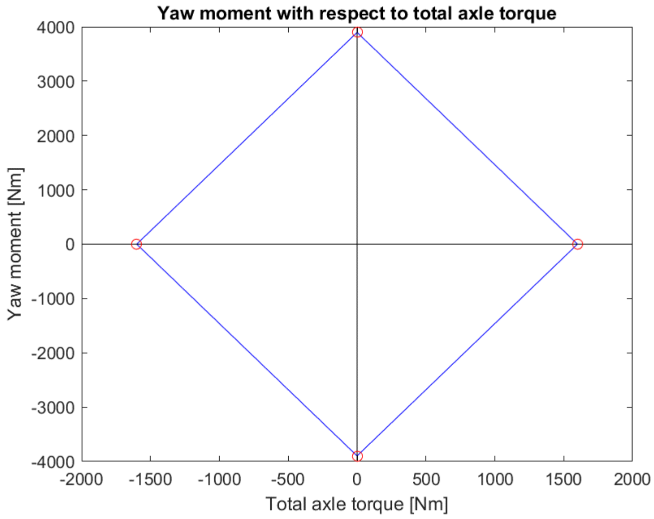


Figure 6. Yaw moment with respect to the total axle torque.

More specifically, from Figure 6 it is possible to see the behaviour of the yaw moment with respect to the total axle torque. When the total axle torque is either the maximum or the minimum value the only yaw moment possible is zero, whilst when the total axle torque is equal to zero, the yaw moment can be either the maximum or the minimum value.

2 Rigid vehicle configuration

A TV NMPC strategy needs the definition of an internal model to predict the future behaviour of the system thus providing the best corrective control action to minimise the internal cost function.

2.1 Internal model

In Figure 7 the four-wheel vehicle model that will be used as reference for the equations in the following section are shown. A non-linear 7-degree-of-freedom (7DOF) vehicle dynamics has been used in this internal model, which includes the longitudinal, lateral and yaw dynamics, as well as the rotation of the four wheels. All symbols have been defined in List of symbols section.

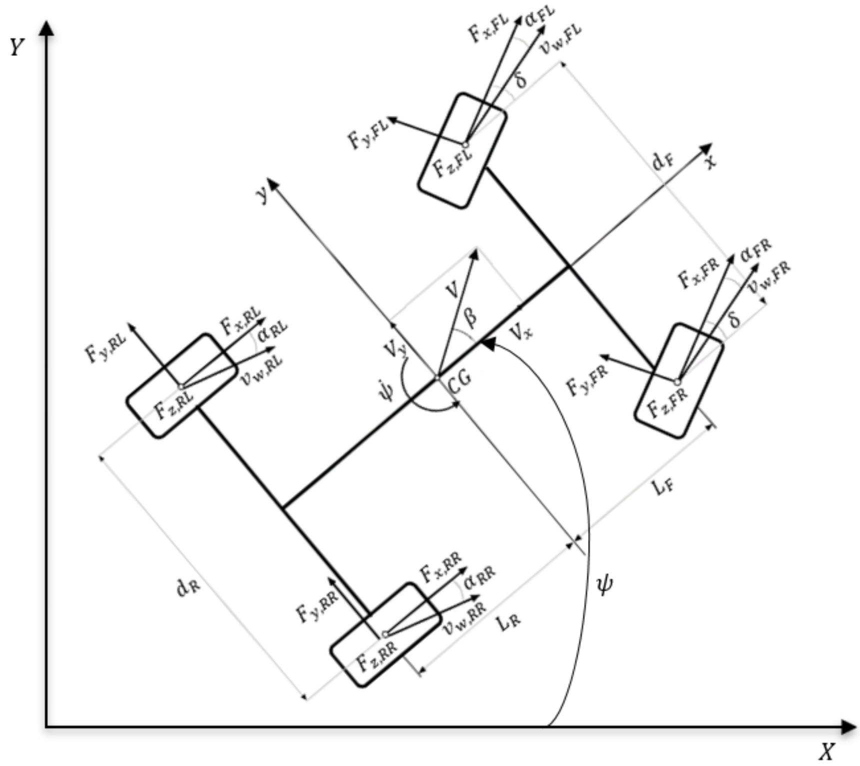


Figure 7. Four-wheel vehicle model.

The equations of motion (EOM) for the four-wheel vehicle model with front wheel steering are:

- **Force balance equation – Longitudinal direction**

$$m\dot{V}_x = \left[\left(F_{x,FL} + F_{x,FR}\right) cos(\delta) - \left(F_{y,FL} + F_{y,FR}\right) sin(\delta) + \left(F_{x,RL} + F_{x,RR}\right) - \frac{1}{2}\rho V^2 S C_x\right] + mV_y\dot{\psi} \qquad (11)$$

where on the left side there is m, which is the mass of the vehicle and $\dot{V}_x$ which is the first derivative of the vehicle longitudinal velocity at its CG and on the right side there are the forces of each tyre projected on the longitudinal direction and the contribution due to the aerodynamic force, where ρ is the density of the air V is the vehicle speed at its CG, S is the vehicle frontal area, C_x is the aerodynamic drag coefficient, and at last V_y is the vehicle lateral speed and $\dot{\psi}$ is the yaw rate.

- **Force balance equation – Lateral direction**

$$m\dot{V}_y = \left[\left(F_{x,FL} + F_{x,FR}\right) sin(\delta) + \left(F_{y,FL} + F_{y,FR}\right) cos(\delta) + \left(F_{y,RL} + F_{y,RR}\right)\right] - mV_x\dot{\psi} \quad (12)$$

where on the left side there is $\dot{V}_y$ which is the first derivative of the vehicle lateral velocity at its CG and, on the right side of the equation there are the forces of each tyre projected on the lateral direction, and then there is V_x which is the longitudinal velocity of the vehicle at its CG multiplied by $\dot{\psi}$ which is the first derivative of vehicle yaw angle (i.e. yaw rate) at its CG.

- **Yaw moment balance equation**

$$J_z\ddot{\psi} = L_F\left[\left(F_{y,FL} + F_{y,FR}\right) cos\, \delta + \left(F_{x,FL} + F_{x,FR}\right) sin\, \delta\right] - L_R\left(F_{y,RL} + F_{y,RR}\right) + \frac{d_F}{2}\left(\left(F_{y,FL} - F_{y,FR}\right)\, \sin(\delta) + \left(F_{x,FR} - F_{x,FL}\right)\cos(\delta)\right) + \frac{d_R}{2}\left(F_{x,RR} - F_{x,RL}\right) \quad (13)$$

where on the left side there is the vehicle moment of inertia J_z multiplied by the second derivative of the yaw angle $\ddot{\psi}$ (i.e. yaw acceleration) of the vehicle at its CG and, on the right side there are the moments generated by the forces on each tyre.

- **Wheel moment balance equation**

$$I_{w,i}\dot{\omega}_{ij} = \tau_{ij} - F_{x,ij}R - fF_{z,ij}R \quad (14)$$

where indexes i defines the front or rear axle, and j the left or the right side of the vehicle; then on the left side of the formula, there is the wheel moment of inertia $I_{w,i}$ of each wheel about its axis of rotation which it is different between front and rear wheels, multiplied by the first derivative of the wheel angular velocity $\dot{\omega}_{ij}$ and, on the right side there is τ_{ij} which is the torque on each wheel, f is the rolling resistance coefficient and $F_{z,ij}$ is the vertical load on each tyre. The total axle torque given by the sum of the torque on each front wheel is:

$$\tau_{tot} = \sum_{j=L,R} \tau_{Fj} \quad (15)$$

- **Forces and slip**

The $F_{x,ij}$ and $F_{y,ij}$ tyre forces are calculated as function of the tyre slip with a simplified Pacejka's Magic Formula. In particular, the resultant tyre force coefficient is obtained as a function of the resultant slip at each tyre from the MF [1]:

$$\mu_{ij}\left(s_{ij}\right) = MF\left(s_{ij}\right) = D\, sin(C\, tan^{-1}(Bs_{ij})) \quad (16)$$

where i is the index for the front or rear axle of the vehicle, j is the index for the left or right side of the vehicle, D_{ij} is the peak factor, C and B_{ij} are respectively the shape and the stiffness factor of the MF. Coefficient C is constant; whilst coefficient D_{ij} can be written in function of the vertical load for each tyre as follows [2]:

$$D_{ij} = a_1 F_{z,ij}^2 + a_2 F_{z,ij} \tag{17}$$

where $F_{z,ij}$ is the vertical load on each tyre and a_1, a_2 are coefficients for tyre formula which include the load influence. To calculate B_{ij}, we define the parameter $(BCD)_{ij}$ called stiffness, which can be calculated as follows:

$$(BCD)_{ij} = a_3\, sin\left(a_4\, tan^{-1}\left(a_5 F_{z,ij}\right)\right) \tag{18}$$

where a_3, a_4, a_5 are coefficients for tyre formula which include the load influence.
At last the stiffness factor B_{ij} is found by dividing the stiffness BCD_{ij} by the shape factor C and the peak factor D_{ij} [2]:

$$B_{ij} = \frac{a_3\, sin\left(a_4\, tan^{-1}\left(a_5 F_{z,ij}\right)\right)}{CD_{ij}} \tag{19}$$

Then there is s_{ij} which is the resultant tyre slip calculated as follows:

$$s_{ij} = \sqrt{s_{x,ij}^2 + s_{y,ij}^2} \tag{20}$$

where $s_{x,ij}$ *and* $s_{y,ij}$ are respectively the longitudinal and lateral slip in driving condition on each tyre which can be calculated with the following formulas:

$$s_{x,ij} = \frac{\omega_{ij} R_e - v_{w,ij} \cos\alpha_{ij}}{v_{w,ij} \cos\alpha_{ij}} \tag{21}$$

$$s_{y,ij} = -\tan\alpha_{ij} \tag{22}$$

where $v_{w,ij}$ is the wheel ground contact point velocity, ω_{ij} is the angular velocity of the driven/braked wheel, R_e is the effective wheel radius for free rolling at zero slip angle, and α_{ij} is the slip angle of each wheel. The wheel ground contact point velocity $v_{w,ij}$ for each wheel is expressed as follows:

$$v_{w,FL} = V - \dot{\psi}\left(\frac{d_F}{2}\cos\beta - L_F \sin\beta\right) \tag{23}$$

$$v_{w,FR} = V + \dot{\psi}\left(\frac{d_F}{2}\cos\beta + L_F \sin\beta\right) \tag{24}$$

$$v_{w,RL} = V - \dot{\psi}\left(\frac{d_R}{2}\cos\beta + L_R \sin\beta\right) \tag{25}$$

$$v_{w,RR} = V + \dot{\psi}\left(\frac{d_R}{2}\cos\beta - L_R \sin\beta\right) \tag{26}$$

As shown in Figure 7, the slip angle is the angle between the direction in which a wheel is pointing (steering direction) and the direction in which the wheel is moving $v_{w,ij}$.
Thus, knowing the velocities at the wheel ground contact point, the four tyre slip angles can be easily derived geometrically and are given by [2]:

$$\alpha_{FL} = tan^{-1}\left(\frac{V_y + \dot{\psi}L_F}{V_x - \dot{\psi}\frac{d_F}{2}}\right) - \delta \tag{27}$$

$$\alpha_{FR} = tan^{-1}\left(\frac{V_y + \dot{\psi}L_F}{V_x + \dot{\psi}\frac{d_F}{2}}\right) - \delta \tag{28}$$

$$\alpha_{RL} = tan^{-1}\left(\frac{V_y - \dot{\psi}L_R}{V_x - \dot{\psi}\frac{d_R}{2}}\right) \tag{29}$$

$$\alpha_{RR} = tan^{-1}\left(\frac{V_y - \dot{\psi}L_R}{V_x + \dot{\psi}\frac{d_R}{2}}\right) \tag{30}$$

where δ is the steering wheel angle, V_y and V_x are respectively the vehicle lateral velocity and vehicle longitudinal velocity, then there is $\dot{\psi}$ which is the first derivative of the yaw angle of the vehicle at its CM, L_F and L_R are respectively the longitudinal distance of the front and rear axle from the CM and d_R, d_F are the total distance between left and right wheel on the rear and on the front axle respectively. Thus, the sideslip angle β of the vehicle at its CM is defined as:

$$\beta = \tan^{-1}\left(\frac{V_y}{V_x}\right) \tag{31}$$

Then using the friction circle equations:

$$\mu_{x,ij} = \frac{s_{x,ij}}{s_{ij}}\mu_{ij}(s_{ij}) \tag{32}$$

$$\mu_{y,ij} = \frac{s_{y,ij}}{s_{ij}}\mu_{ij}(s_{ij}) \tag{33}$$

the tyre force coefficients $\mu_{x,ij}$ *and* $\mu_{y,ij}$ are expressed in the longitudinal and lateral direction.
Some assumptions must be done before explicating the vertical forces:

1. The static load is evenly distributed between the right and the left side of the vehicle;
2. The values of wheels acceleration are approximated to the vehicle acceleration;
3. Front and rear roll centres are at the same height.

Under these assumptions, it is possible to explicate $F_{z,ij}$ on each of the four wheels which can be calculated by the load distribution formulas:

$$F_{z,FL} = \frac{mgL_R}{2L_{TOT}} - \frac{ma_xh}{2L_{TOT}} - \frac{ma_y}{d_F}\left(\frac{L_RRCH}{L_{TOT}} + \frac{k_{roll,F}H_{roll}}{k_{roll,F} + k_{roll,R}}\right) - \frac{1}{2}\rho V^2SC_x\frac{h}{2L_{TOT}} \tag{34}$$

$$F_{z,FR} = \frac{mgL_R}{2L_{TOT}} - \frac{ma_xh}{2L_{TOT}} + \frac{ma_y}{d_F}\left(\frac{L_RRCH}{L_{TOT}} + \frac{k_{roll,F}H_{roll}}{k_{roll,F} + k_{roll,R}}\right) - \frac{1}{2}\rho V^2SC_x\frac{h}{2L_{TOT}} \tag{35}$$

$$F_{z,RL} = \frac{mgL_F}{2L_{TOT}} + \frac{ma_x h}{2L_{TOT}} - \frac{ma_y}{d_R}\left(\frac{L_F RCH}{L_{TOT}} + \frac{k_{roll,R}H_{roll}}{k_{roll,F} + k_{roll,R}}\right) + \frac{1}{2}\rho V^2 SC_x \frac{h}{2L_{TOT}} \tag{36}$$

$$F_{z,RR} = \frac{mgL_F}{2L_{TOT}} + \frac{ma_x h}{2L_{TOT}} + \frac{ma_y}{d_R}\left(\frac{L_F RCH}{L_{TOT}} + \frac{k_{roll,R}H_{roll}}{k_{roll,F} + k_{roll,R}}\right) + \frac{1}{2}\rho V^2 SC_x \frac{h}{2L_{TOT}} \tag{37}$$

the first term of each equation is the static load of the vehicle where m is the vehicle mass, g is the gravitational acceleration, $L_R$and L_F are respectively the distance of the CM from the rear and the front axle, L_{TOT} is the distance between the rear and the front axle; the second term is the longitudinal load distribution term where a_x is the longitudinal acceleration of the vehicle, defined as follows:

$$a_x = \dot{V}_x - V_y\dot{\psi} \tag{38}$$

where $\dot{V}_x$ is the first derivative of the vehicle longitudinal velocity and V_y is the vehicle lateral velocity of the vehicle at its CG multiplied by the yaw rate of the vehicle.
Then in *(34)*,*(35)*,*(36)* and *(37)* , h is the height of CG from the road surface next term is the lateral load transfer where d_{TOT} is the width of the vehicle as represented in Figure 7, and a_y is the lateral acceleration of the vehicle which can be calculated as follows:

$$a_y = \dot{V}_y + V_x\dot{\psi} \tag{39}$$

where $\dot{V}_y$ is the first derivative of the vehicle lateral velocity at its CG, V_x is the longitudinal vehicle The third term of the vertical load is composed by two terms: the first one is related to the load transfers through the rigid links of the suspensions whilst the second one is related to the load transfers through the suspension springs and anti-roll bars. RCH is the height of the roll centre from the road surface which, for assumption, is the same for the front and the rear and H_{roll} is the distance between CG and the roll centre. The last term is the vertical load transfer due to the aerodynamic forces.
Therefore, the longitudinal and lateral tyre forces are then given by the following formulas:

$$F_{x,ij} = \mu_{x,ij}F_{z,ij} \tag{40}$$

$$F_{y,ij} = \mu_{y,ij}F_{z,ij} \tag{41}$$

- **Power losses**

Once the forces are defined, it is possible to calculate the tyre slip power loss of each wheel, in the longitudinal and lateral directions, with the following formulas:

$$P_{loss,xslip,ij} = F_{x,ij}v_{slip,x,ij} \tag{42}$$

$$P_{loss,yslip,ij} = F_{y,ij}v_{slip,y,ij} \tag{43}$$

where $v_{slip,x,ij}$ is the longitudinal slip speed, which is given by:

$$v_{slip,x,ij} = \omega_{ij}R_e - v_{w,ij}\cos\alpha_{ij} \tag{44}$$

$$v_{slip,y,ij} = v_{w,ij}\cos\alpha_{ij}\, s_{y,ij} \tag{45}$$

where , the term $v_{w,ij} \cos \alpha_{ij}$ is the longitudinal velocity of each wheel, ω_{ij} is the angular velocity of the driven/braked wheel R_e is the effective wheel radius for free rolling at zero slip angle and $v_{slip,y,ij}$ is the lateral slip velocity which can be computed with Eq. (45).
The vehicle resultant velocity which can be computed as follows:

$$V = \sqrt{V_x^2 + V_y^2} \tag{46}$$

the total tyre slip power loss in both directions, are defined as follows:

$$P_{loss,xslip}^{tot} = \sum_{ij} P_{loss,xslip,ij} \tag{47}$$

$$P_{loss,yslip}^{tot} = \sum_{ij} P_{loss,yslip,ij} \tag{48}$$

2.2 Driveability controller

The developments include the implementation of a regenerative braking controller based on the following scheme:

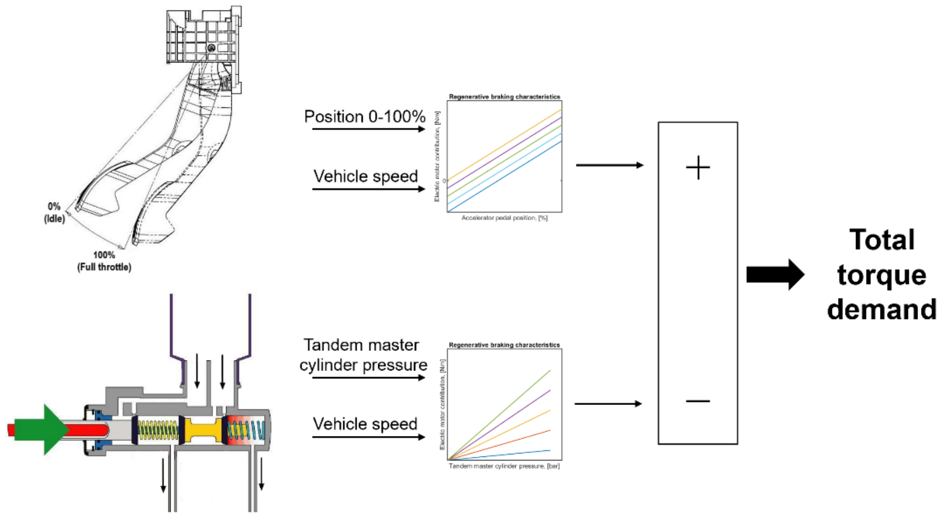


Figure 8. Regenerative braking system scheme

In SYS2WHEEL, the vehicle demonstrator will be characterised by a conventional friction braking system layout, with standard brake booster, tandem master cylinder and stability control unit. Therefore, the regenerative braking controller will have to work around the constraints of the physically available braking system, without using seamless brake blending algorithms, typical of electrified vehicles including brake-by-wire systems, e.g., electro-hydraulic or electro-mechanical braking systems.

Using the driveability map based on the accelerator pedal position and on the vehicle speed it is possible to calculate the accelerator-pedal-related part of the total regenerative braking torque demand. More specifically, for low accelerator pedal positions (e.g., in the range 0-20%), the electric vehicle has the regenerative braking system active, thus a negative torque is provided. Nevertheless, for very low speed values the regenerative braking torque value in the map is set to zero, considered the very low values of energy that can be regenerated in these conditions, and in order to prevent the vehicle from moving backwards. Moreover, for flexibility of implementation, the specific regenerative braking controller includes an additional map that considers the tandem master cylinder pressure and the vehicle speed. The pressure value is measured by a sensor and it is proportional to the driver's force on the brake pedal and, therefore, on the basis of the pressure measured in the master cylinder it is possible to add more regenerative braking torque, and modulate it according to the driver's input on the brake pedal.

Lastly, the two regenerative braking torque contributions are added together in order to find the total torque demand to be applied to the vehicle through the electric powertrains in braking conditions.

Regarding the interactions between the NMPC controlling the electric powertrains and the friction brakes present on the demonstrator vehicles, the idea is to use the flag variable from the conventional production ABS unit of the SYS2WHEEL vehicle in order to reduce the regenerative braking contribution specified by the SYS2WHEEL NMPC, if interventions of the conventional ABS occur. The literature already includes a multitude of patents (e.g., see FR2972411A, Renault; WO12108001A1, Toyota; JP2011031698A, Hitachi; KR20110139836A, Mando; US6231134B, Advics, Aisin Seiki, Toyota; US5318355A, Honda) proposing similar solutions for the management of the regenerative braking contribution during the interventions of the ABS actuated through the friction brakes.

Obviously, such limitation will be much less of a concern during the interventions of the wheel slip control limitation function in traction, despite the conventional stability control unit installed on the vehicle could still give origin to some form of interference, by actuating interventions of the friction brakes to limit wheel slip in traction. In SYS2WHEEL, the constraint on the individual wheel slip incorporated within the NMPC will work as main traction control function.

2.3 Nonlinear model predictive control - optimal control problem formulation

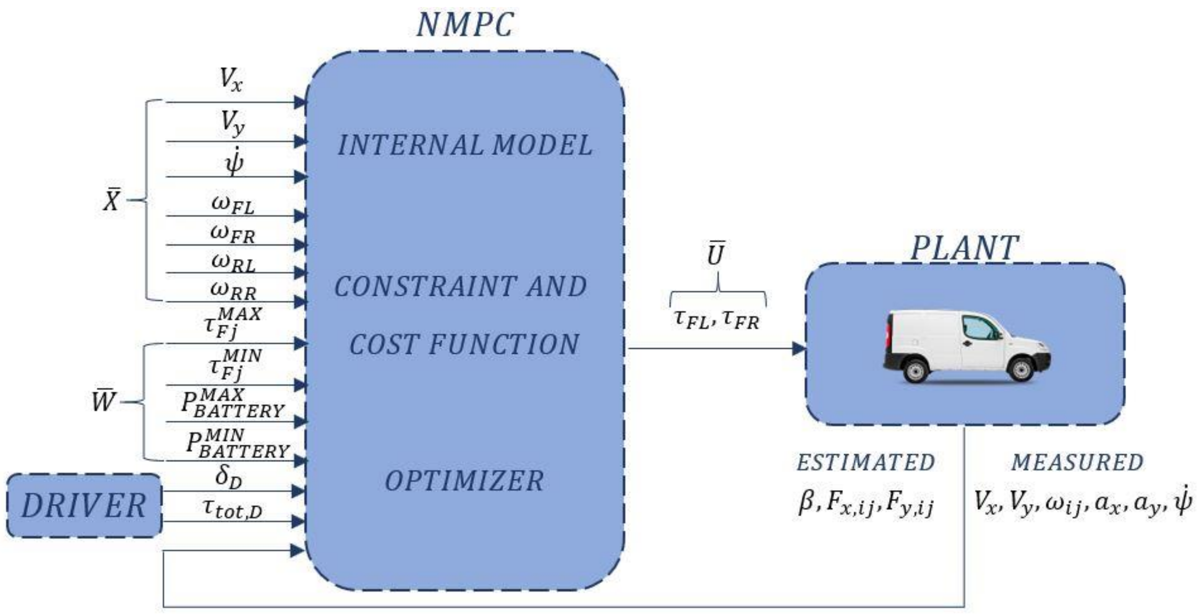


Figure 9. Schematic diagram for NMPC and plant – vehicle configuration.

The Nonlinear Model Predictive Control (NMPC) approach is employed in this study to develop the integrated control structure. The controller is intended to track the desired yaw rate for handling improvement [3] and ensure the energy efficiency by minimizing both lateral and longitudinal tyre slip power losses and the powertrain power losses. To implement controller ACADO Toolkit was used. ACADO Toolkit is a software environment and algorithm collection written in C++ for automatic control and dynamic optimization [5] which has been used to solve the constrained nonlinear optimization problem in the NMPC. ACADO Toolkit generates a C-code, which is then usable in Simulink.
A vehicle model in state-space form is developed by combining the vehicle dynamics equations *(11)*, *(12)*,*(13)* and the wheel dynamics equation (14) and the full vehicle model in standard state-space form is expressed as:

$$\dot{X} = f(X(t), W(t), U(t)) \tag{49}$$

where X is the state vector defined as follows :

$$X = [V_x, V_y, \dot{\psi}, \omega_{FL}, \omega_{FR}, \omega_{RL}, \omega_{RR}]^T \tag{50}$$

and its elements are defined in the section above.

W is the parameters vector defined as follows :

$$W = [\tau_{Fj}^{min}, \tau_{Fj}^{max}, P_{BATTERY}^{min}, P_{BATTERY}^{max}]^T \tag{51}$$

where $\tau_{Fj}^{min}, \tau_{Fj}^{max}$ are the minimum and the maximum value of torque respectively on both front sides of the vehicle and $P_{BATTERY}^{min}, P_{BATTERY}^{max}$ are the minimum and the maximum value of the battery power respectively. All these parameters will be used in the computation of the optimal control input.

U is the controlled input vector defined as follows :

$$U = [\tau_{FL}, \tau_{FR}]^T \quad (52)$$

The discrete-time state-space form of the vehicle model can be derived by discretization of (24) and thus can be written as follows:

$$X^{k+1} = f\ (X^k, W^k, U^k) \quad (53)$$

Some assumptions must be done before developing the NMPC controller: the driver inputs such as steer angles, torques on wheels, vehicle speed and the load transfer are constant during prediction horizon N which is a common approach in developing the NMPC for vehicle stability control.

Chosen parameters for the vehicle dynamic are the yaw rate and total traction torque ($\dot{\psi}$,$\tau_{tot,tr}$).

Chosen parameters for the vehicle power efficiency are the tyre slip power loss in lateral and longitudinal direction, and the power losses $P_{loss,yslip}^{tot}$, $P_{loss,xslip}^{tot}$, $P_{loss,EM}$, $P_{loss,INV}$.

Therefore, to find the optimal control input a constrained quadratic optimization problem has to be solved by using the following cost function [5]:

$$J = \frac{1}{2}\left\|Z_V^N - Z_{V,d}^N\right\|_{Q_x}^2 + \frac{1}{2}\sum_{k=0}^{N-1}\left(\left\|Z_V^k - Z_{V,d}^k\right\|_{Q_x}^2 + \|U^k\|_R^2\right) \quad (54)$$

where Z_V is the output vector defined as:

$$Z_V = [\tau_{tot}, \dot{\psi}, s_\alpha, P_{loss,xslip}^{tot}, P_{loss,yslip}^{tot}, P_{loss,PWT}]^T \quad (55)$$

where the firsts two elements are defined in the previous section, s_α is the slack variable used in the rear sideslip angle constraint, and $P_{loss,PWT}$ is defined by Eq. *(9)* .

Then there is $Z_{V,d}$which is the output vector with the desirable values defined as:

$$Z_{V,d} = [\tau_{tot,D}, \dot{\psi}_d, 0\ , 0\ , 0\ , 0]^T \quad (56)$$

where the first element is $\tau_{tot,D}$ which is the driver torque demand, the second one is the desirable yaw rate which is computed using a look-up table.

This look-up table is generated by simulating the passive vehicle configuration while cornering at constant speed. Hence, several simulations were made with varying steering angle and constant speed and in the end, the following map is created:

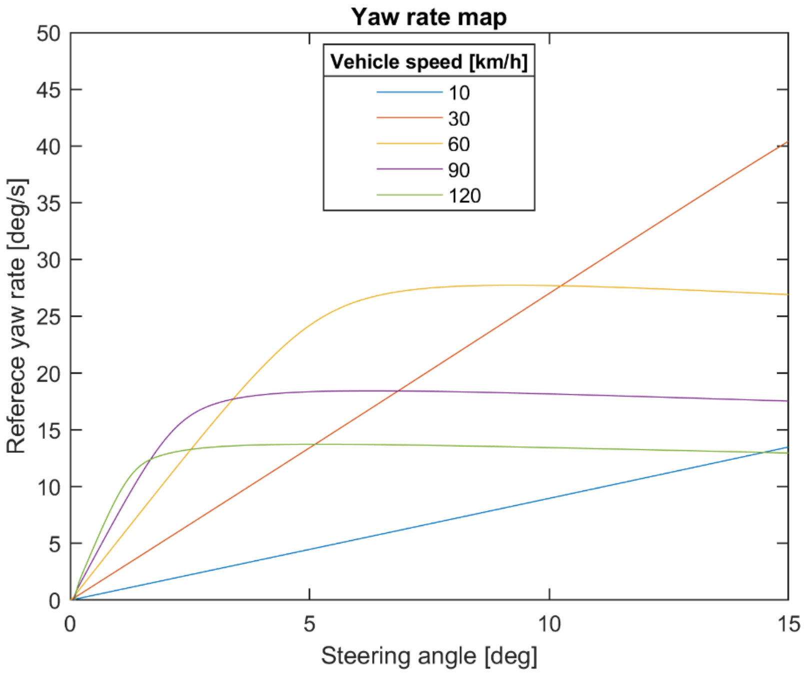


Figure 10. Yaw rate map.

The others are all zero because it is the desirable value of either the slack variable for the rear slip angle and the power losses.Q_x and R are positive semi-definite the weight matrices. Then the aim is to minimize the cost function and model predictive control offers a method for incorporating both an objective as well as constraints. Taken into the context of the stabilization problem, the objective can be leveraged to express the driver's intended vehicle behaviour while the constraints represent the physical limitations of the vehicle. [4] The first constraint is on τ_{Fj}^{k} and considers the limitation of the maximum torque and the maximum tyre force capacity according to the friction between tyres and the road. [3] The other two constraints limit the value of the rear sideslip angle and of battery power level.
The first constraint can be written as follows:

$$-\min\left(F_{x,ij}^{max}R, \tau_{Fj}^{min}\right) \leq \tau_{Fj}^{k} \leq \min\left(F_{x,ij}^{max}R, \tau_{Fj}^{max}\right) \qquad (57)$$

where $\tau_{Fj}^{min}$and τ_{Fj}^{max} are the minimum and the maximum torque available at each front wheel respectively and defined as positives values, and $F_{x,ij}^{max}$ is the maximum value of the longitudinal force expressed as follows:

$$F_{x,ij}^{max} = \mu_{x,ij}F_{z,ij}(0)\sqrt{1-\left(\frac{F_{y,ij}(0)}{\mu_{y,ij}F_{z,ij}(0)}\right)^{2}} \qquad (58)$$

where $F_{y,ij}(0)$ and $F_{z,ij}(0)$ are the lateral tyre force and the vertical tyre force at the beginning of the horizon, respectively.
Therefore, the NMPC problem can be formulated as follows:

$$arg_U min\, J\ (Z_V^k, U^k) \tag{59}$$

$$s.t.:\ \ Z^{k+1} = f\ (X^k, U^k, W^k) \tag{60}$$

$$-\min\left(F_{x,ij}^{max} R, \tau_{Fj}^{min}\right) \leq \tau_{Fj}^k \leq \min\left(F_{x,ij}^{max} R, \tau_{Fj}^{max}\right) \tag{61}$$

$$s_\alpha \geq 0 \tag{62}$$

$$-\alpha_{Rj}^{max}(1 + s_\alpha) \leq \alpha_{Rj}^k \leq \alpha_{Rj}^{max}(1 + s_\alpha) \tag{63}$$

$$P_{BATTERY}^{min} \leq P_{BATTERY}^k \leq P_{BATTERY}^{max} \tag{64}$$

Further developments will include the tuning of energy efficiency functions and tuning of wheel slip control functions of the controller.

2.4 Selection of results

To assess the controller behaviour a single step steer manoeuvre with different vehicle speed was performed.

2.4.1 Manoeuvre

- Step steer at 80 km/h with maximum steering angle of 10 deg
- Step steer at 120 km/h with maximum steering angle of 10 deg

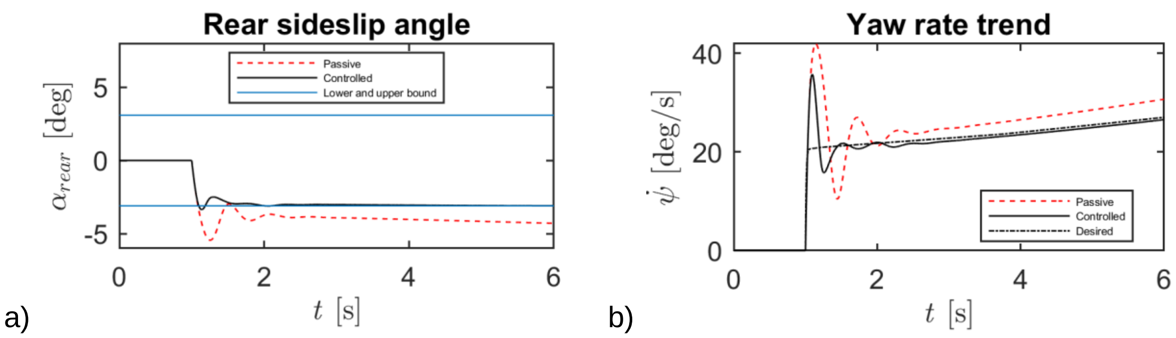


Figure 11. Step steer manoeuvre rear sideslip angle soft constrained at 80 km/h.

In Figure 11 it is shown the behaviour of the controlled rigid vehicle when both yaw rate tracking term and slack variable are taken into account in the cost function. As it is clearly visible at

~ 1 s the rear sideslip angle exceeds the boundary fixed and the controller intervention acts to correct this undesirable behaviour. At the same time a good yaw rate tracking is shown.

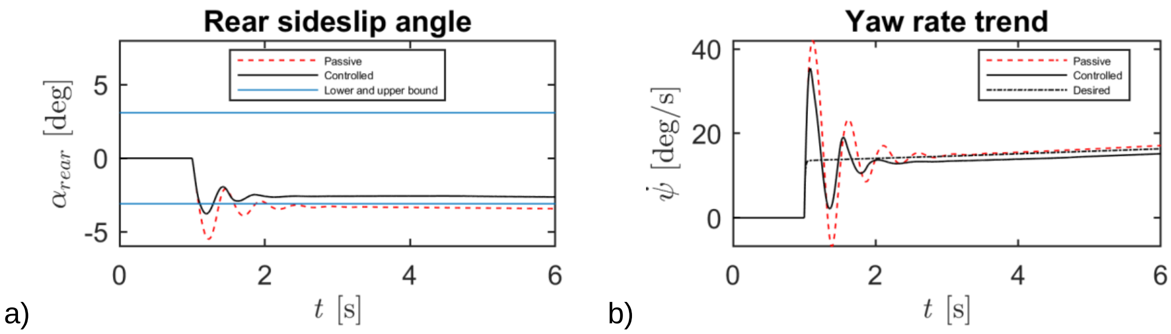


Figure 12. Step steer manoeuvre yaw rate tracking in emergency conditions at 120 km/h.

In Figure 12 it is shown the behaviour of the controlled rigid vehicle in emergency conditions with a step steer at 120 km/h. More in detail the yaw rate oscillation is highly damped and the rear sideslip angle is constrained between the thresholds.

3 Articulated vehicle configuration

3.1 Internal model mathematical derivation

In Figure 13 the articulated vehicle model that will be used as reference for the equations in section 3.1 is shown. A non-linear 8-degree-of-freedom (8DOF) articulated vehicle dynamics has been used in this internal model, which includes the longitudinal, lateral, vehicle yaw and trailer yaw, as well as the rotation of the four wheels of the vehicle. All symbols have been defined in the list of symbols.

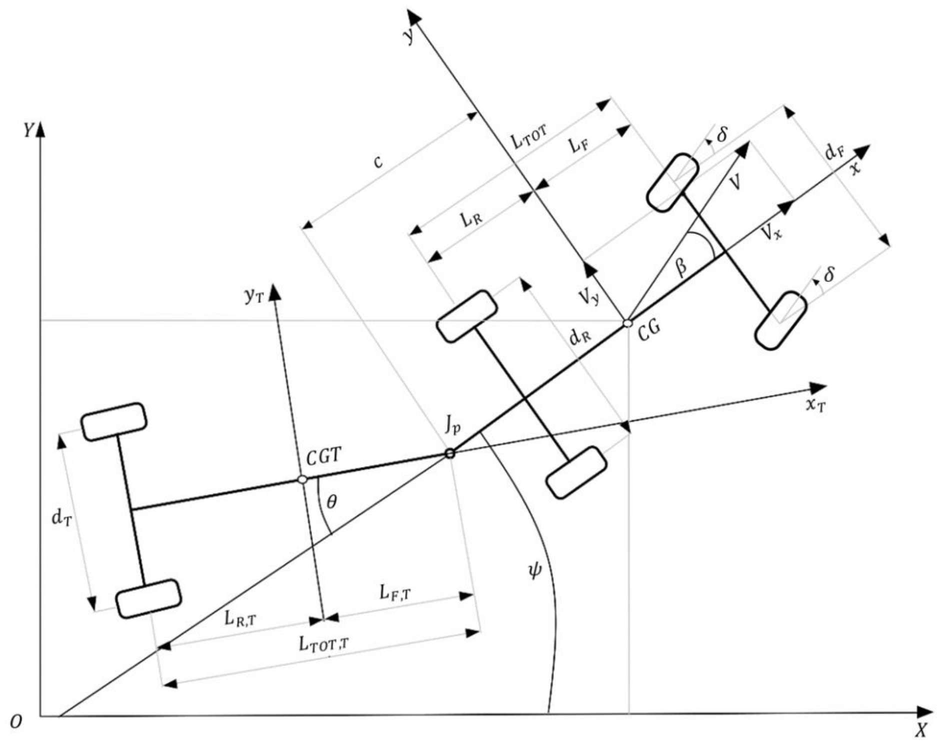


Figure 13. Articulated vehicle model.

3.1.1 Trailer lateral and longitudinal acceleration

The first step to calculate the acceleration of the trailer is to express the velocity in the centre of mass of the towing vehicle to the velocity in the hitch joint.
By considering:

$$V_{CG} = V \tag{65}$$

V_J is the velocity in the hitch joint of the articulated vehicle is expressed as follows:

$$V_J = V_{CG} + \dot{\psi} \times (J_p - CG) \tag{66}$$

where V_{CG} is the velocity in the centre of mass of the towing vehicle, $\dot{\psi}$ is the yaw rate of the towing vehicle and $(J_p - CG)$ is the distance between the hitch joint and the centre of mass of the towing vehicle.

$$(J_p - CG) = \sqrt{\left(X_{CG} - X_{J_p}\right)^2 + \left(Y_{CG} - Y_{J_p}\right)^2} \tag{67}$$

The coordinates of the hitch joint in the inertial frame which origins are in J_p, are:

$$X_{J_p} = 0 \ \& \ Y_{J_p} = 0 \tag{68}$$

While the coordinates, in the same reference frame, of the centre of gravity of the towing vehicle in the inertial frame are:

$$X_{CG} = c\cos(\psi) \ \& \ Y_{CG} = c\sin(\psi) \tag{69}$$

By performing the vector product:

$$\dot{\psi} \times (J_p - CG) = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \dot{\psi} \\ -c\cos(\psi) & -c\sin(\psi) & 0 \end{bmatrix} = \left(\dot{\psi}c\sin(\psi)\right)\hat{\imath} - \left(\dot{\psi}c\cos(\psi)\right)\hat{\jmath} \tag{70}$$

$$V_{CG} = \left(V_x\cos(\psi) - V_y\sin(\psi)\right)\hat{\imath} + \left(V_x\sin(\psi) + V_y\cos(\psi)\right)\hat{\jmath} \tag{71}$$

The result of the summation of the two relative velocities provides us the velocity in the hitch joint:

$$V_{J_p} = \left(V_x\cos(\psi) - V_y\sin(\psi) + \dot{\psi}c\sin(\psi)\right)\hat{\imath} + \left(V_x\sin(\psi) + V_y\cos(\psi) - \dot{\psi}c\cos(\psi)\right)\hat{\jmath} \tag{72}$$

The velocity in the centre of gravity of the trailer is obtained as follows:

$$\left(CGT - J_p\right) = \left(-L_{F,T}\cos(\psi - \theta)\right)\hat{\imath} + \left(-L_{F,T}\sin(\psi - \theta)\right)\hat{\jmath} \tag{73}$$

$$V_{CGT} = V_{J_p} + (\dot{\psi} - \dot{\theta}) \times (CGT - J_p) \tag{74}$$

$$\begin{aligned}(\dot{\psi} - \dot{\theta}) \times (CGT - J_p) &= \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \dot{\psi} - \dot{\theta} \\ -L_{F,T}\cos(\psi - \theta) & -L_{F,T}\sin(\psi - \theta) & 0 \end{bmatrix} \\ &= \left(\left(\dot{\psi} - \dot{\theta}\right)\left(L_{F,T}\sin(\psi - \theta)\right)\right)\hat{\imath} + \left(-L_{F,T}\cos(\psi - \theta)(\dot{\psi} - \dot{\theta})\right)\hat{\jmath}\end{aligned} \tag{75}$$

By resolving the previous steps, we are finally able to find the velocity in the centre of mass of the trailer:

$$\begin{aligned}V_{CGT} = &\left(V_x\cos(\psi) - V_y\sin(\psi) + \dot{\psi}c\sin(\psi) + \left(\dot{\psi} - \dot{\theta}\right)L_{F,T}\sin(\psi - \theta)\right)\hat{\imath} \\ &+ \left(V_x\sin(\psi) + V_y\cos(\psi) - \dot{\psi}c\cos(\psi) - L_{F,T}\cos(\psi - \theta)\left(\dot{\psi} - \dot{\theta}\right)\right)\hat{\jmath}\end{aligned} \tag{76}$$

To find the lateral and longitudinal acceleration of the trailer in the inertial reference frame XY system the time derivatives of the lateral and longitudinal components of the velocity in the centre of gravity of the trailer have been computed:

$$V_{x,CGT} = V_x\cos(\psi) - V_y\sin(\psi) + \dot{\psi}c\sin(\psi) + \left(\dot{\psi} - \dot{\theta}\right)L_{F,T}\sin(\psi - \theta) \tag{77}$$

$$V_{y,CGT} = V_x \sin(\psi) + V_y \cos(\psi) - \dot{\psi} c \cos(\psi) - L_{F,T} \cos(\psi - \theta) \left(\dot{\psi} - \dot{\theta}\right) \tag{78}$$

By performing the time derivates to find the lateral and the longitudinal acceleration in the inertial reference frame:

$$\begin{aligned} a_{xT,XY,CGT} = {} & \dot{V}_x \cos(\psi) - V_x \sin(\psi)\,\dot{\psi} - \dot{V}_y \sin(\psi) - V_y \cos(\psi)\,\dot{\psi} + \ddot{\psi} c sin(\psi) + \dot{\psi}^2 c \cos(\psi) \\ & + \left(\ddot{\psi} - \ddot{\theta}\right) L_{F,T} \sin(\psi - \theta) + \left(\dot{\psi} - \dot{\theta}\right)^2 L_{F,T} \cos(\psi - \theta) \end{aligned} \tag{79}$$

$$\begin{aligned} a_{yT,XY,CGT} = {} & \dot{V}_x \sin(\psi) + V_x \cos(\psi)\,\dot{\psi} + \dot{V}_y \cos(\psi) \\ & - V_y \sin(\psi)\dot{\psi} - \ddot{\psi} c \cos(\psi) + \dot{\psi}^2 c \sin(\psi) + L_{F,T} \sin(\psi - \theta) \left(\dot{\psi} - \dot{\theta}\right)^2 \\ & - L_{F,T} \cos\,(\psi - \theta)(\ddot{\psi} - \ddot{\theta}) \end{aligned} \tag{80}$$

By pre multiplying the previous expressions of the accelerations of the trailer by the rotation matrix R_T the expressions of the trailer's accelerations in its reference frame are obtained.

$$R_T = \begin{bmatrix} \cos\,(\psi - \theta) & \sin(\psi - \theta) \\ -\sin\,(\psi - \theta) & \cos\,(\psi - \theta) \end{bmatrix} \tag{81}$$

$$a_{xT,CGT} = \cos(\psi - \theta)\, a_{xT,XY,CGT} + \sin\,(\psi - \theta) a_{yT,XY,CGT} \tag{82}$$

$$a_{yT,CGT} = -\sin(\psi - \theta)\, a_{xT,XY,CGT} + \cos\,(\psi - \theta)\, a_{yT,XY,CGT} \tag{83}$$

3.1.2 Kinetic energy derivation

The coordinates of the CGT of the trailer are defined with the generalized coordinates and then by deriving the velocity on the CGT is obtained. Then writing the kinematic energy of the system and by deriving opportunely it is possible to write the left side of the EOM.

The Kinetic energy of the system is:

$$\mathrm{T} = \frac{1}{2} m V_{CG}^2 + \frac{1}{2} m_T V_{CGT}^2 + \frac{1}{2} J_z \dot{\psi}^2 + \frac{1}{2} J_T (\dot{\psi} - \dot{\theta})^2 \tag{84}$$

where m, m_T, J_z, J_T, V_{CG} and V_{CGT} are respectively the masses, the barycentric moments of inertia about an axis perpendicular to the road and the velocities of the towing vehicle in its CG and the trailer in its CGT in inertial frame which are expressed respectively as in *(71)* and *(76)*.

The velocity in the centre of mass of both towing vehicle and the trailer are:

$$V_{CG}^2 = V_x^2 + V_y^2 \tag{85}$$

$$\begin{aligned} V_{CGT}^2 = {} & V_x^2 + V_y^2 + \dot{\psi}^2 c^2 + \left(\dot{\psi} - \dot{\theta}\right)^2 L_{F,T}^2 - 2V_x \left(\dot{\psi} - \dot{\theta}\right) L_{F,T} \sin(\theta) - 2V_y \dot{\psi} c - 2V_y \dot{\psi} c \\ & - 2V_y \left(\dot{\psi} - \dot{\theta}\right) L_{F,T} \cos(\theta) + 2\dot{\psi} c (\dot{\psi} - \dot{\theta}) L_{F,T} \cos\,(\theta) \end{aligned} \tag{86}$$

By inserting the velocities in kinetic energy expression, it follows:

$$\begin{aligned} \mathrm{T} = {} & \frac{1}{2} M \left(V_x^2 + V_y^2\right) + \frac{1}{2} \dot{\psi}^2 J_1(\theta) + \frac{1}{2} \dot{\theta}^2 J_3 - \dot{\psi} \dot{\theta} J_2(\theta) - m_T V_y \left(c\dot{\psi} + L_{F,T} \left(\dot{\psi} - \dot{\theta}\right) \cos(\theta)\right) \\ & - m_T V_x (\dot{\psi} - \dot{\theta}) L_{F,T} \sin\,(\theta) \end{aligned} \tag{87}$$

where $J_1(\theta)$, $J_2(\theta)$ and J_3 are mass moment of inertia defined as follows:

$$M = m + m_T \tag{88}$$

$$J_1(\theta) = J_Z + J_T + m_T\left(L_{F,T}^2 + c^2 + 2L_{F,T}c\cos(\theta)\right) \tag{89}$$

$$J_2(\theta) = J_T + m_T\left(L_{F,T}^2 + L_{F,T}c\cos(\theta)\right) \tag{90}$$

$$J_3 = J_T + m_T L_{F,T}^2 \tag{91}$$

where J_z and J_T are the baricentric moment of inertia about an axis perpendicular to the road of the vehicle and of the trailer, respectively.

3.1.3 Equations of motion

The equations of motion for the articulated vehicle model are obtained through Lagrange equations, in order to proceed with this approach, four generalized coordinates are chosen: X and Y which are the inertial coordinates of the CG of the vehicle , ψ which is the yaw angle of the vehicle and the last one is the hitch angle θ.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial F}{\partial \dot{q}_i} = Q_i \tag{92}$$

where T is the kinetic energy of the system, q_i are the generalized coordinates, F is the Rayleight dissipation function with the damping coefficient Γ that is associated to the hinge between tractor and trailer and the Q_i that are the generalized forces.

The equations of motion are obtained as in [7]-[9] :

$$A\frac{\partial}{\partial t}\left(\left\{\frac{\partial \mathrm{T}}{\partial w}\right\}\right) + \dot{A}\left\{\frac{\partial \mathrm{T}}{\partial w}\right\} - \left\{\frac{\partial \mathrm{T}}{\partial q_k}\right\} - \left[w^T A^T \frac{\partial A}{\partial q_k}\right]\left\{\frac{\partial \mathrm{T}}{\partial w}\right\} = Q_i \tag{93}$$

$$\frac{\partial}{\partial t}\left(\left\{\frac{\partial \mathrm{T}}{\partial w}\right\}\right) + A^T\left(\dot{A} - \left[w^T A^T\left(\frac{\partial A}{\partial q_k}\right)\right]\right)\left\{\frac{\partial \mathrm{T}}{\partial w}\right\} - A^T\left\{\frac{\partial \mathrm{T}}{\partial q_k}\right\} = A^T Q_i \tag{94}$$

where w is the vector containing the generalized velocities $V_x, V_y, \dot{\psi}$ and $\dot{\theta}$

$$w = \left[V_x\ V_y\ \dot{\psi}\ \dot{\theta}\right]^T \tag{95}$$

The vector $\dot{q}$ instead is the vector containing the derivatives of the generalized coordinates.

$$\dot{q} = \left[\dot{X}\ \dot{Y}\ \dot{\psi}\ \dot{\theta}\right]^T \tag{96}$$

Matrix A is the rotation matrix which is used to make the passage from inertial frame to the towing vehicle reference frame.

$$A = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 & 0 \\ \sin(\psi) & \cos(\psi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{97}$$

The transposed of matrix A is A^T and the time derivative of A is $\dot{A}$.

The state equations are written with the reference to generalized velocities that are not simply the derivatives of the generalized coordinates. Often it is expedient to use as generalized velocities suitable combination of the derivatives of the coordinates.

The velocities $V_x, V_y, \dot{\psi}$ and $\dot{\theta}$ are linked to the derivatives of generalized coordinates $\dot{X}, \dot{Y}, \dot{\psi}$ and $\dot{\theta}$ by the relationship below:

$$w = A^T \dot{q} \tag{98}$$

The matrix form of the previous relationship is:

$$\begin{bmatrix} V_x \\ V_y \\ \dot{\psi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 & 0 \\ -\sin(\psi) & \cos(\psi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{\psi} \\ \dot{\theta} \end{bmatrix} \tag{99}$$

where we consider the generalized coordinates X, Y, ψ and θ and Q_i are the corresponding generalized forces F_X, F_Y and the moments related to rotation ψ and θ. F is the Rayleigh dissipation function that is associated to the hinge between the towing vehicle and trailer. Γ si the damping coefficient.

The Rayleight dissipation function is:

$$F = \frac{1}{2}\Gamma\dot{\theta}^2 \tag{100}$$

By solving Eq. *(94)*, the equations of motions in the towing vehicle reference frame are obtained:

- **1° EOM**

$$M(\dot{V}_x - \dot{\psi}V_y) - m_T L_{F,T}(\ddot{\psi} - \ddot{\theta})\sin(\theta) - 2m_T L_{F,T}\cos(\theta)\,\dot{\theta}\dot{\psi} + m_T L_{F,T}\cos(\theta)\,\dot{\theta}^2 + m_T\dot{\psi}^2\big(c + L_{F,T}\cos(\theta)\big) = Q_x \tag{101}$$

- **2° EOM**

$$M(\dot{V}_y + V_x\dot{\psi}) - m_T\ddot{\psi}\big(c + L_{F,T}\cos(\theta)\big) + m_T L_{F,T}\ddot{\theta}\cos(\theta) - m_T L_{F,T}\sin(\theta)\,\big(\dot{\psi} - \dot{\theta}\big)^2 = Q_y \tag{102}$$

- **3° EOM**

$$J_1(\theta)\ddot{\psi} - J_2(\theta)\ddot{\theta} + m_T L_{F,T} c\sin(\theta)\,\big(\dot{\theta}^2 - 2\dot{\theta}\dot{\psi}\big) - m_T L_{F,T}\sin(\theta)\,\big(\dot{V}_x - V_y\dot{\psi}\big) - m_T\big(\dot{V}_y + V_x\dot{\psi}\big)\big(c + L_{F,T}\cos(\theta)\big) = Q_\psi \tag{103}$$

- **4° EOM**

$$J_3\ddot{\theta} - J_2(\theta)\ddot{\psi} + m_T L_{F,T}\cos(\theta)\,\big(\dot{V}_y + V_x\dot{\psi}\big) + m_T L_{F,T}\sin(\theta)\,\Big(\dot{V}_x - \dot{\psi}\big(V_y - c\dot{\psi}\big)\Big) = Q_\theta - \Gamma\dot{\theta} \tag{104}$$

- **Generalized forces derivation**

Generalized forces rigid vehicle

The first step is to write the velocity of the centre $P_{i,j}$ of the contact area of the $i,j = (Front, Rear; Left, Right)$ wheel, located in a point whose coordinates are $x_{i,j}$ and $y_{i,j}$ in the reference frame of the vehicle. Thus, $V_{pi,j}$ is defined as follows:

$$V_{pi,j} = V_{CG} + \dot{\psi} \times (P_{i,j} - CG) \tag{105}$$

The velocity of the centre of mass of the towing vehicle is expressed in *(71)*:

$$\dot{\psi} \times (P_{i,j} - CG) = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \dot{\psi} \\ x_{i,j} & y_{i,j} & 0 \end{bmatrix} = -y_{i,j}\,\dot{\psi}\hat{\imath} + x_{i,j}\,\dot{\psi}\hat{\jmath} \tag{106}$$

$$V_{pi,j} = (V_x - y_{i,j}\dot{\psi})\hat{\imath} + (V_y + x_{i,j}\dot{\psi}\,)\hat{\jmath} \tag{107}$$

By pre multiplying the components of the previous velocity by δt the virtual displacement of each wheel in the reference frame of the vehicle is obtained:

$$\begin{cases} \delta_{pi,jx} = \delta x - y_{i,j}\delta\psi \\ \delta_{pi,jy} = \delta y + x_{i,j}\delta\psi \end{cases} \tag{108}$$

By considering a steering angle only on front wheels δ the forces that the i,j-th wheel exerts in the reference frame of the towing vehicle are:

$$\begin{cases} F_{xV} = (F_{x,FL} + F_{x,FR})\,cos(\delta) - (F_{y,FL} + F_{y,FR})\,sin(\delta) + (F_{x,RL} + F_{x,RR}) \\ F_{yV} = (F_{x,FL} + F_{x,FR})\,sin(\delta) + (F_{y,FL} + F_{y,FR})\,cos(\delta) + (F_{y,RL} + F_{y,RR}) \end{cases} \tag{109}$$

By executing the scalar product between the forces exerted by the wheels and the corresponding virtual displacements we can obtain the virtual work of the rigid vehicle:

$$\begin{aligned} \delta L_v = {} & \Big((F_{x,FL} + F_{x,FR})\,cos(\delta) - (F_{y,FL} + F_{y,FR})\,sin(\delta) + (F_{x,RL} + F_{x,RR}) \Big)\,\delta x \\ & + \Big((F_{x,FL} + F_{x,FR})\,sin(\delta) + (F_{y,FL} + F_{y,FR})\,cos(\delta) + (F_{y,RL} + F_{y,RR}) \Big)\,\delta y \\ & + \Bigg(L_F\big[(F_{y,FL} + F_{y,FR})\,cos\,\delta + (F_{x,FL} + F_{x,FR})\,sin\,\delta\big] - L_R(F_{y,RL} + F_{y,RR}) \\ & + \frac{d_F}{2}\big((F_{y,FL} - F_{y,FR})\,\sin(\delta) + (F_{x,FR} - F_{x,FL})\cos(\delta)\big) \\ & + \frac{d_R}{2}(F_{x,RR} - F_{x,RL}) \Bigg)\,\delta\psi \end{aligned} \tag{110}$$

Generalized forces and virtual work trailer

The coordinates of point $P_{j,T}$, centre of the contact zone of the $j = (Left, Right)$ wheel of the trailer, in the inertial frame are:

$$\begin{cases} X_{p,j,T} = X - c\cos(\psi) - L_{TOT,T}\cos(\psi-\theta) - y_{j,T}\sin(\psi-\theta) \\ Y_{p,j,T} = Y - c\sin(\psi) - L_{TOT,T}\sin(\psi-\theta) + y_{j,T}\cos(\psi-\theta) \end{cases} \quad (111)$$

The previous terms are the coordinates of the centre of the contact zone of each wheel in inertial frame, where $y_{j,T}$ is the y-coordinate of the wheel in the trailer not-inertial frame.

The velocity is obtained by performing the time derivative:

$$\dot{X}_{p,j,T} = \dot{X} + c\sin(\psi)\,\dot{\psi} + L_{TOT,T}\sin(\psi-\theta)\left(\dot{\psi}-\dot{\theta}\right) - y_{j,T}\cos(\psi-\theta)(\dot{\psi}-\dot{\theta}) \quad (112)$$

$$\dot{Y}_{p,j,T} = \dot{Y} - c\cos(\psi)\,\dot{\psi} - L_{TOT,T}\cos(\psi-\theta)\left(\dot{\psi}-\dot{\theta}\right) - y_{j,T}\sin(\psi-\theta)(\dot{\psi}-\dot{\theta}) \quad (113)$$

To express the velocity of the point $P_{j,T}$ in the reference frame of the trailer the rotation matrix is used:

$$\begin{bmatrix} \dot{X}_{p,j,T,R} \\ \dot{Y}_{p,j,T,R} \end{bmatrix} = \begin{bmatrix} \cos(\psi-\theta) & \sin(\psi-\theta) \\ -\sin(\psi-\theta) & \cos(\psi-\theta) \end{bmatrix} \begin{bmatrix} \dot{X}_{p,j,T} \\ \dot{Y}_{p,j,T} \end{bmatrix} \quad (114)$$

By explaining the calculations:

$$\dot{X}_{p,j,T,R} = \dot{X}(\cos(\psi)\cos(\theta) + \sin(\psi)\sin(\theta)) + \dot{Y}(\sin(\psi)\cos(\theta) - \sin(\theta)\cos(\psi)) + c\dot{\psi}\sin(\theta) - y_{j,T}(\dot{\psi}-\dot{\theta}) \quad (115)$$

$$\dot{Y}_{p,j,T,R} = -\dot{X}\sin(\psi-\theta) + \dot{Y}\cos(\psi-\theta) - c\dot{\psi}(\sin(\psi-\theta)\sin(\psi) + \cos(\psi-\theta)\cos(\psi)) - L_{TOT,T}(\dot{\psi}-\dot{\theta}) \quad (116)$$

By using the components of velocity on the centre of mass of the towing vehicle in *(71)* it is possible to obtain:

$$V_{x,p,j,T,R} = V_x\cos(\theta) - V_y\sin(\theta) + c\dot{\psi}\sin(\theta) - y_{j,T}(\dot{\psi}-\dot{\theta}) \quad (117)$$

$$V_{y,p,j,T,R} = V_x\sin(\theta) + V_y\cos(\theta) - c\dot{\psi}\cos(\theta) - L_{TOT,T}(\dot{\psi}-\dot{\theta}) \quad (118)$$

The virtual displacements, to determinate the virtual work, are:

$$\{\delta s\} = \{\delta x, \delta y, \delta\psi, \delta\theta\}^T \quad (119)$$

where the firsts two are virtual linear displacements and lasts two are virtual angular displacements. By multiplying both sides of velocities equations by the time δt we automatically obtain the expressions of virtual displacement in the reference frame of trailer:

$$\delta_{xT_{Pj}} = \delta x cos(\theta) - \delta y sin(\theta) + c\delta\psi\sin(\theta) - y_{j,T}(\delta\psi - \delta\theta) \quad (120)$$

$$\delta_{yT_{Pj}} = \delta x sin(\theta) + \delta y cos(\theta) - c\delta\psi\cos(\theta) - L_{TOT,T}(\delta\psi - \delta\theta) \quad (121)$$

The forces of j-th wheel in the reference frame $x_T y_T z_T$ without considering the steering angle of the wheel of the trailer are : $\sum_{j=(L,R)} F_{x,j,T}$ and $\sum_{j=(L,R)} F_{y,j,T}$. Where j indicates the left and right wheel of the trailer.

By doing the scalar product between the forces defined previously and the virtual displacement in the not-inertial frame of the trailer it is possible to obtain the virtual work of the trailer:

$$\delta L_T = \left(\sum_{j=(L,R)} \left[F_{x,j,T}\cos(\theta) + F_{y,j,T}\sin(\theta)\right] \right)\delta x + \left(\sum_{j=(L,R)} \left[-F_{x,j,T}\sin(\theta) + F_{y,j,T}\cos(\theta)\right] \right)\delta y$$
$$+ \left(\sum_{j=(L,R)} \left\{F_{x,j,T}\left[csin(\theta) - y_{j,T}\right] + F_{y,j,T}\left[-ccos(\theta) - L_{TOT,T}\ \right]\right\} \right)\delta\psi \qquad (122)$$
$$+ \left(\sum_{j=(L,R)} \left\{F_{x,j,T}y_{j,T} + F_{y,j,T}L_{TOT,T}\right\} \right)\delta\theta$$

- **Articulated vehicle virtual work**

$$\delta L = \sum_{i=x,y} F_i\delta_i + \sum_{j=\psi,\theta} M_j\delta_j \qquad (123)$$

where F_i are the resultant forces acting at the road-wheel interface on the towing vehicle not-inertial frame on the x and y direction. M_j is the resultant moment about, the centre of mass of the towing vehicle for the yaw angle ψ and the total moment about, the hitch joint for the hitch angle θ. F_i and M_j have been obtained with the scalar product between the forces exert by the wheels of the articulated vehicle and the respective virtual displacements.
The last step is to determinate the total virtual work by summing the virtual work of insulated vehicle and the virtual work of the trailer:

$$\delta L = \delta L_V + \delta L_T \qquad (124)$$

The complete expression of the virtual work is:

$$\delta L = \Bigg[\left(F_{x,FL} + F_{x,FR}\right) cos(\delta) - (F_{y,FL} + F_{y,FR})\, sin(\delta) + \left(F_{x,RL} + F_{x,RR}\right) + \sum_{j=(L,R)} \left[F_{x,j,T} \cos(\theta) + F_{y,j,T} \sin(\theta)\right]\Bigg] \delta x + \Bigg[\left(F_{x,FL} + F_{x,FR}\right) sin(\delta) + \left(F_{y,FL} + F_{y,FR}\right) cos(\delta) + (F_{y,RL} + F_{y,RR}) + \sum_{j=(L,R)} \left[-F_{x,j,T} \sin(\theta) + F_{y,j,T} \cos(\theta)\right]\Bigg] \delta y + \Bigg[L_F\left[\left(F_{y,FL} + F_{y,FR}\right) cos\,\delta + \left(F_{x,FL} + F_{x,FR}\right) sin\,\delta\right] - L_R\left(F_{y,RL} + F_{y,RR}\right) + \frac{d_F}{2}\left((F_{y,FL} - F_{y,FR})\ \sin(\delta) + \left(F_{x,FR} - F_{x,FL}\right)\cos(\delta)\right) + \frac{d_R}{2}\left(F_{x,RR} - F_{x,RL}\right) + \sum_{j=(L,R)} \left\{F_{x,j,T}\left[csin(\theta) - y_{j,T}\right] + F_{y,j,T}\left[-ccos(\theta) - L_{TOT,T}\ \right]\right\}\Bigg] \delta\psi + \Bigg[\sum_{j=(L,R)} \left\{F_{x,j,T} y_{j,T} + F_{y,j,T} L_{TOT,T}\right\}\Bigg] \delta\theta \tag{125}$$

By differentiating the virtual work with respect to the different virtual displacements it is possible to obtain the total generalized forces due to the i-th wheel of the articulated vehicle.

$$Q_x = \frac{\partial \delta L}{\partial \delta x} = \left(F_{x,FL} + F_{x,FR}\right) cos(\delta) - (F_{y,FL} + F_{y,FR})\, sin(\delta) + \left(F_{x,RL} + F_{x,RR}\right) + \sum_{j=(L,R)} \left[F_{x,j,T} \cos(\theta) + F_{y,j,T} \sin(\theta)\right] - \frac{1}{2}\rho V^2 S C_x \tag{126}$$

In the previous generalized force, we are also considering the influence of the aerodynamic drag term.

$$Q_y = \frac{\partial \delta L}{\partial \delta y} = \left(F_{x,FL} + F_{x,FR}\right) sin(\delta) + \left(F_{y,FL} + F_{y,FR}\right) cos(\delta) + (F_{y,RL} + F_{y,RR}) + \sum_{j=(L,R)} \left[-F_{x,j,T} \sin(\theta) + F_{y,j,T} \cos(\theta)\right] \tag{127}$$

$$Q_\psi = \frac{\partial \delta L}{\partial \delta \psi} = L_F\left[\left(F_{y,FL} + F_{y,FR}\right) cos\,\delta + \left(F_{x,FL} + F_{x,FR}\right) sin\,\delta\right] - L_R\left(F_{y,RL} + F_{y,RR}\right) + \frac{d_F}{2}\left((F_{y,FL} - F_{y,FR})\ \sin(\delta) + \left(F_{x,FR} - F_{x,FL}\right)\cos(\delta)\right) + \frac{d_R}{2}\left(F_{x,RR} - F_{x,RL}\right) + \sum_{j=(L,R)} \left\{F_{x,j,T}\left[c \sin(\theta) - y_{j,T}\right] + F_{y,j,T}\left[-c \cos(\theta) - L_{TOT,T}\ \right]\right\} \tag{128}$$

$$Q_\theta = \frac{\partial \delta L}{\partial \delta \theta} = \sum_{j=(L,R)} \{F_{x,j,T} y_{j,T} + F_{y,j,T} L_{TOT,T}\} \quad (129)$$

Then on the right side of the EOM there are the forces and moments, which are the composition of the vehicle contribution and the trailer contribution.
Therefore, the EOM are written by neglecting the aerodynamical forces and assuming α_T, which is the road grade of inclination, equals to zero.

- ## Final expression of the equation of motion
- **Force balance equation – Longitudinal direction of the vehicle**

$$M(\dot{V}_x - \dot{\psi} V_y) - m_T L_{F,T}(\ddot{\psi} - \ddot{\theta}) \sin(\theta) - 2 m_T L_{F,T} \dot{\psi} \dot{\theta} \cos(\theta) + m_T L_{F,T} \dot{\theta}^2 \cos(\theta) + m_T \dot{\psi}^2 (c + L_{F,T} \cos(\theta)) = (F_{x,FL} + F_{x,FR}) cos(\delta) - (F_{y,FL} + F_{y,FR}) sin(\delta) + (F_{x,RL} + F_{x,RR}) + \sum_{j=(L,R)} [F_{x,j,T} \cos(\theta) + F_{y,j,T} \sin(\theta)] - \frac{1}{2} \rho V^2 S C_x \quad (130)$$

where $M = (m + m_T)$ is the total mass of the vehicle and trailer , m_T is the mass of the trailer, $\dot{V}_x$ is the time derivative of the x-component of the vehicle velocity at its CM, θ is the hitch angle which is the angle between the longitudinal direction of the vehicle and the longitudinal direction of the trailer, $\dot{\theta}$ and $\ddot{\theta}$ are its first and second time derivative, $\dot{\psi}$ and $\ddot{\psi}$ are the first derivative and second derivative of the yaw angle which is the angle between the longitudinal direction of the vehicle and the X-axis absolute frame, $L_{F,T}$ is the distance between the hinge and the CM of the trailer, c is the distance between the hinge and the CM of the tractor and δ is the steering wheel angle of the vehicle. On the right side of the equation there are the summation of forces of the vehicle and the trailer respectively, projected on the longitudinal direction and the contribution of the aerodynamic force on the vehicle.

- **Force balance equation – Lateral direction of the vehicle**

$$\begin{aligned} M(\dot{V}_y + \dot{\psi} V_x) - m_T \ddot{\psi}(c + L_{F,T} \cos(\theta)) + m_T L_{F,T} \ddot{\theta} \cos(\theta) - m_T L_{F,T} \sin(\theta) (\dot{\psi} - \dot{\theta})^2 \\ = (F_{x,FL} + F_{x,FR}) sin(\delta) + (F_{y,FL} + F_{y,FR}) cos(\delta) + (F_{y,RL} + F_{y,RR}) \\ + \sum_{j=(L,R)} [-F_{x,j,T} \sin(\theta) + F_{y,j,T} \cos(\theta)] \end{aligned} \quad (131)$$

where $\dot{V}_y$ is the time derivative of the y-component of the vehicle velocity at its CM and the rest of the symbols used in this equation are the same described for Eq. *(102)*. The forces on the right side are projected on the lateral direction.

- **Yaw moment balance equation of the vehicle**

$$\begin{aligned} J_1(\theta)\ddot{\psi} - J_2(\theta)\ddot{\theta} + m_T L_{F,T} c \sin(\theta)\left(\dot{\theta}^2 - 2\dot{\theta}\dot{\psi}\right) - m_T L_{F,T}\sin(\theta)\left(\dot{V}_x - V_y\dot{\psi}\right) \\ - m_T\left(\dot{V}_y + V_x\dot{\psi}\right)\left(c + L_{F,T}\cos(\theta)\right) \\ = L_F\left[\left(F_{y,FL} + F_{y,FR}\right)\cos\delta + \left(F_{x,FL} + F_{x,FR}\right)\sin\delta\right] - L_R\left(F_{y,RL} + F_{y,RR}\right) \\ + \frac{d_F}{2}\left(\left(F_{y,FL} - F_{y,FR}\right)\sin(\delta) + \left(F_{x,FR} - F_{x,FL}\right)\cos(\delta)\right) + \frac{d_R}{2}\left(F_{x,RR} - F_{x,RL}\right) \\ + \sum_{j=(L,R)}\left\{F_{x,j,T}\left[c\sin(\theta) - y_{j,T}\right] + F_{y,j,T}\left[-c\cos(\theta) - L_{TOT,T}\right]\right\} \end{aligned} \quad (132)$$

- **Yaw moment balance equation of the trailer about the hinge**

$$\begin{aligned} J_3\ddot{\theta} - J_2(\theta)\ddot{\psi} + m_T L_{F,T}\cos(\theta)\left(\dot{V}_y + V_x\dot{\psi}\right) + m_T L_{F,T}\sin(\theta)\left(\dot{V}_x - \dot{\psi}\left(V_y - c\dot{\psi}\right)\right) \\ = \sum_{j=(L,R)}\left\{F_{x,j,T}\, y_{j,T} + F_{y,j,T} L_{TOT,T}\right\} - \Gamma\dot{\theta} \end{aligned} \quad (133)$$

- **Wheel moment balance equation**

$$I_w \dot{\omega}_{ij} = \tau_{ij} - F_{x,ij}R - fF_{z,ij}R \quad (134)$$

where I_w is the wheel moment of inertia $\dot{\omega}_{ij}$ is the angular acceleration of each wheel, $\tau_{ij}$is the torque on each wheel ,$(-F_{x,ij}R)$ is the torque due to the longitudinal force on each wheel of the vehicle and $fF_{z,ij}R$ is the rolling resistance term.

- **Forces and slip**

The $F_{x,ij}$, $F_{y,ij}$, $F_{x,ij,T}$, $F_{y,ij,T}$ tyre forces are calculated as function of the tyre slip with Pacejka's Magic Formula as seen for the rigid vehicle model. Also, for the slip it is possible to use the formulation seen for the rigid vehicle.

The sideslip angles of the wheels of the vehicle are the same as for the vehicle model Eq. *(27)-(30)* and in a similar way it is possible to write the sideslip angles of the wheels of the trailer as follows:

$$\alpha_{j,T} = \tan^{-1}\left[\frac{V_{y,p,j,T,R}}{V_{x,p,j,T,R}}\right] \quad (135)$$

By substituting the velocity of the centre of contact area, it follows:

$$\alpha_{j,T} = \tan^{-1}\left[\frac{V_x\sin\theta + V_y\cos\theta - c\dot{\psi}\cos\theta - L_{TOT,T}\left(\dot{\psi} - \dot{\theta}\right)}{V_x\cos\theta - V_y\sin\theta + c\dot{\psi}\sin\theta - y_{j,T}\left(\dot{\psi} - \dot{\theta}\right)}\right] \quad (136)$$

where j is the index for the left and right side of the trailer and all the other factors are defined in the above equations.

In the calculation of $F_{z,ij}$ for the nonlinear model, some assumptions must be done before explicating the vertical forces:

1. The static load of the vehicle is evenly distributed between the right and the left side;
2. The values of wheels acceleration are approximated to the vehicle acceleration;
3. Front and rear roll centres are at the same height.

Therefore, the generic vertical load transfer can be expressed as:

$$F_{z,ij,k} = \frac{F_{z,ij,static}}{2} + \frac{F_{z,ij,longit.}}{2} + k_1 \Delta F_{z,a_y,ij,k} + k_2 \frac{\Delta F_{z,aero,ij,k}}{2} \quad (137)$$

where $i = F, R$ is the index for the front, rear axle of the vehicle, $j = R, L$ is the index of the right and left side of the vehicle, $k = T$ is the index used to identify the trailer factors, k_1 is a coefficient and can be -1 if we are referring to the left side of the vehicle or 1 if we are referring to the right side of the vehicle and k_2 is a coefficient and can be -1 if we are referring to the front of the vehicle, +1 if we are referring to the rear of the vehicle and is equal to 0 if we are referring to the trailer $(k = T)$.

3.1.4 Vertical loads of the articulated vehicle

- **Vertical Loads – Static balance**

Towing Vehicle

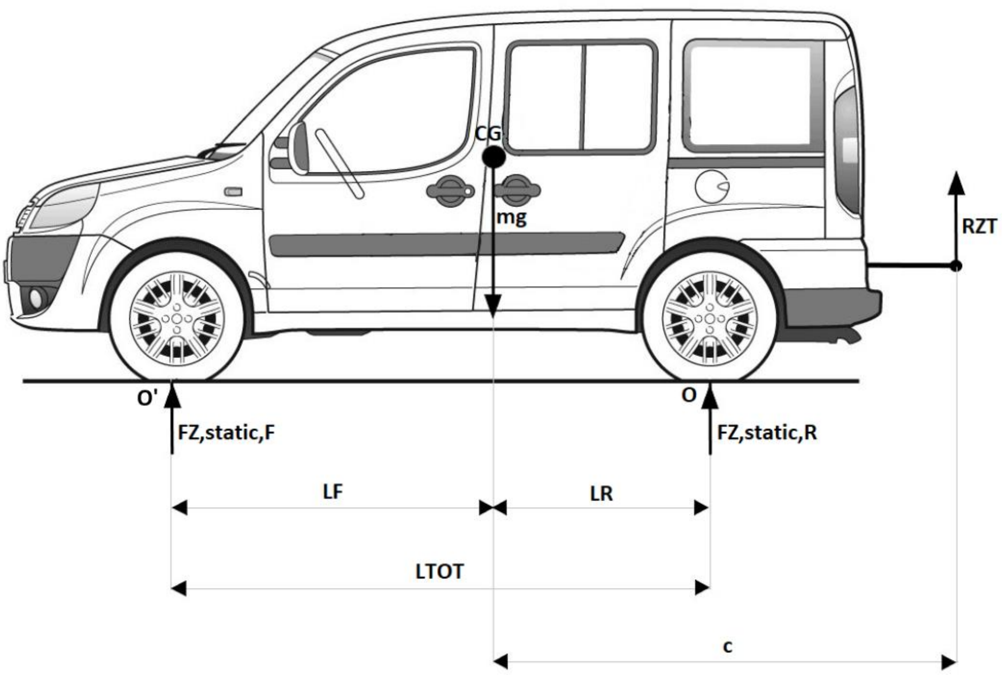


Figure 14. Static balance - free body diagram towing vehicle

The first analysis to do is about the towing vehicle. To solve its static balance, the moment balance about point O on the rear of the vehicle must be executed:

$$F_{Z,STATIC,F} L_{TOT} = mgL_R + R_{ZT}(c - L_R) \quad (138)$$

$$F_{Z,STATIC,F} = \frac{mgL_R + R_{ZT}(c - L_R)}{L_{TOT}} \quad (139)$$

By performing the moment balance about the point O' it is possible to find the $F_{Z,STATIC,R}$:

$$F_{Z,STATIC,R}L_{TOT} = mgL_F - R_{ZT}(c + L_F) \quad (140)$$

$$F_{Z,STATIC,R} = \frac{mgL_F - R_{ZT}(c + L_F)}{L_{TOT}} \quad (141)$$

Trailer

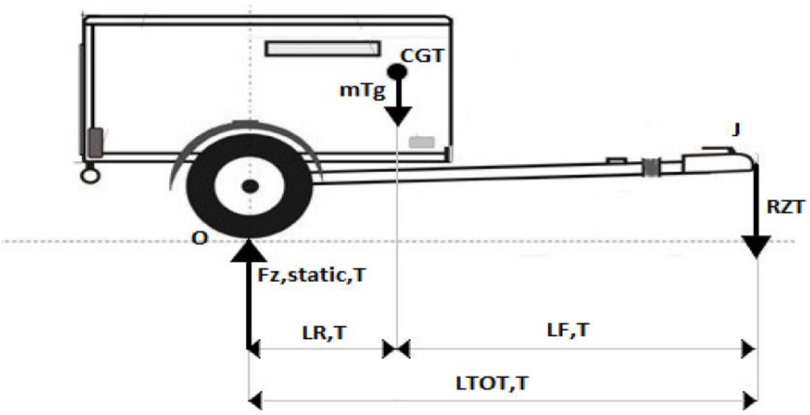


Figure 15. Static balance- free body diagram trailer.

The value of R_{ZT} is obtainable by performing the moment balance about point O of the trailer:

$$R_{ZT}L_{TOT,T} = -m_T g L_{R,T} \quad (142)$$

$$R_{ZT} = \frac{-m_T g L_{R,T}}{L_{TOT,T}} \quad (143)$$

From the moment balance about the point J we can find $F_{Z,STATIC,T}$:

$$F_{Z,STATIC,T}L_{TOT,T} = m_T g L_{F,T} \quad (144)$$

$$F_{Z,STATIC,T} = \frac{m_T g L_{F,T}}{L_{TOT,T}} \quad (145)$$

Articulated vehicle

It is now possible to write down the expressions of vertical loads due to the static contribution:

$$F_{Z,STATIC,F} = \frac{mgL_R}{L_{TOT}} - \frac{m_T g L_{R,T}(c - L_R)}{L_{TOT,T}L_{TOT}} \quad (146)$$

$$F_{Z,STATIC,R} = \frac{mgL_F}{L_{TOT}} + \frac{m_T g L_{R,T}(L_F + c)}{L_{TOT,T}L_{TOT}} \quad (147)$$

$$F_{Z,STATIC,T} = \frac{m_T g L_{F,T}}{L_{TOT,T}} \quad (148)$$

- **Vertical load - Longitudinal load transfer**

The second fundamental component of the verticals loads are due to the longitudinal load transfer during traction and braking.

<u>Towing vehicle</u>

As in the case of the static components of the vertical load we start to carry out the balances of the moments and of the forces in the towing vehicle.

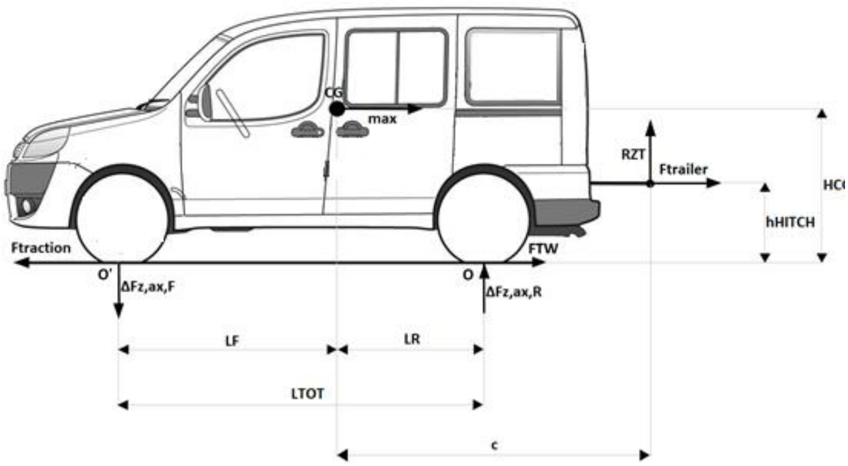


Figure 16. Longitudinal load transfer - free body diagram towing vehicle.

By performing the moment balance about the point O it is possible to obtain the $\Delta F_{z,ax,F}$:

$$\Delta F_{Z,ax,F} L_{TOT} + R_{ZT}(c - L_R) = m a_x HCG + F_{Trailer} h_{HITCH} \quad (149)$$

We are considering a front-wheel drive vehicle so the $F_{traction}$ term is only on the front axle. Moreover, we have inserted a friction term only for completeness reason. This term is almost null.

The term HCG and h_{HITCH} are the height of the centre of mass of the towing vehicle and the height of the hitch joint, respectively.

$$\Delta F_{z,ax,F} = \frac{m a_x HCG + F_{Trailer} h_{HITCH} - R_{ZT}(c - L_R)}{L_{TOT}} \quad (150)$$

By doing the vertical forces balance we obtain:

$$R_{ZT} + \Delta F_{Z,ax,R} = \Delta F_{z,ax,F} \quad (151)$$

The moment about the point O' is:

$$\Delta F_{Z,ax,R} = \frac{m a_x HCG + F_{Trailer} h_{HITCH} - R_{ZT}(c + L_F)}{L_{TOT}} \quad (152)$$

In the same way $\Delta F_{Z,ax,F}$ is:

$$\Delta F_{Z,ax,F} = \frac{ma_x HCG + F_{Trailer} h_{HITCH} - R_{ZT}(c - L_R)}{L_{TOT}} \quad (153)$$

Trailer

In the trailer we are considering two forces on the hitch joint, the inertial force in the centre of gravity, the vertical $\Delta F_{Z,ax,T}$ term and the rolling resistance that is almost null. We can proceed with the balances:

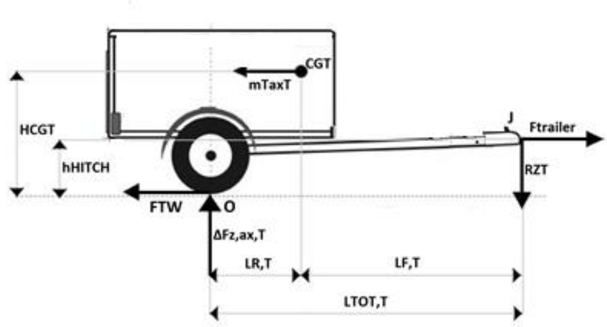


Figure 17. Longitudinal load transfer - free body diagram trailer.

By performing the moment balance about the contact point O between the wheel and the road it follows:

$$R_{ZT} L_{TOT,T} + F_{Trailer} h_{HITCH} = m_T a_{xT} HCGT \quad (154)$$

While the moment balance about the point J is:

$$\Delta F_{Z,ax,T} L_{TOT,T} + F_{TW} h_{HITCH} = m_T a_{xT}(HCGT - h_{HITCH}) \quad (155)$$

The vertical forces balance of the trailer is:

$$R_{ZT} = \Delta F_{Z,ax,T} \quad (156)$$

The longitudinal forces balance of the trailer is:

$$F_{Trailer} = m_T axT + F_{TW} \quad (157)$$

The expression of longitudinal load transfer in the trailer is then:

$$\Delta F_{z,ax,T} = \frac{m_T a_{xT}(HCGT - h_{HITCH}) - F_{TW} h_{HITCH}}{L_{TOT,T}} \quad (158)$$

By performing the opportune substitutions the front and rear component of longitudinal load transfer in the towing vehicle are obtained:

$$\Delta F_{ax,F} = \frac{ma_x HCG}{L_{TOT}} + \frac{(m_T a_{xT} + F_{TW}) h_{HITCH}}{L_{TOT}} - \left(\frac{m_T a_{xT}(HCGT - h_{HITCH}) - F_{TW} h_{HITCH}}{L_{TOT} L_{TOT,T}}\right)(c - L_R) \quad (159)$$

$$\Delta F_{ax,R} = \frac{ma_x HCG}{L_{TOT}} + \frac{(m_T a_{xT} + F_{TW}) h_{HITCH}}{L_{TOT}} - \left(\frac{m_T a_{xT}(HCGT - h_{HITCH}) - F_{TW} h_{HITCH}}{L_{TOT,T} L_{TOT}}\right)(L_F + c) \quad (160)$$

Remembering that the friction resistance is:

$$F_{TW} = fF_Z \qquad (161)$$

where f is the friction coefficient.

- **Vertical load- Lateral load transfer**

The roll centre represents the centre of the instantaneous rotation of the vehicle body relative to the ground and that the suspensions give a roll stiffness K_{roll} we can write the follow expression:

$$K_{roll} = \frac{\Delta M_{antiroll}}{\Delta \phi} \qquad (162)$$

where $\Delta\phi$ is the roll angle variation and $\Delta M_{antiroll}$ is the anti-roll moment.
To calculate the load transfer, a free vehicle body diagram where is simulated the roll motion of the vehicle body is defined. The points CG' and J' are the centre of gravity of the towing vehicle and the hitch joint after a roll motion of the body. As hypothesis the front and rear roll centres are at the same height.
The free body diagram below shows the towing vehicle cornering around a right-hand turn. In this free body diagram it is also considered the lateral contribution of the hitch joint.

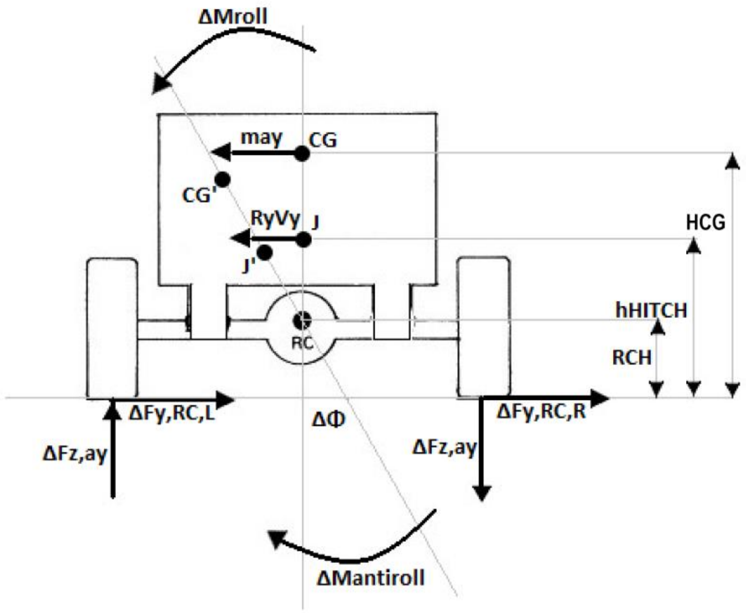


Figure 18. Lateral load transfer - free body diagram towing vehicle rear view.

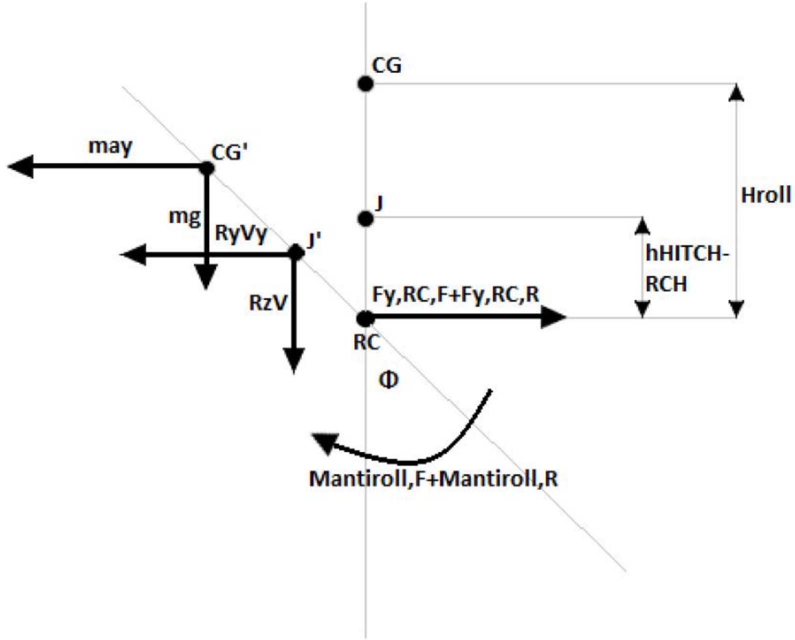


Figure 19. Lateral load transfer - towing vehicle.

By performing the moment balance about RC axis it follows:

$$M_{antiroll,F} + M_{antiroll,R} = mgH_{roll}\sin(\phi) + ma_yH_{roll}\cos(\phi) + R_{yVy}(h_{HITCH} - RCH)\cos(\phi) + R_{zV}(h_{HITCH} - RCH)\sin(\phi) \quad (163)$$

where R_{yVy} is the lateral force contribution of the hitch joint on the towing vehicle, R_{zV} is the vertical force contribution of the hitch joint on the towing vehicle, H_{roll} is the roll centre height and RCH is the height of the roll centre from the road.

The lateral force balance is:

$$ma_y + R_{yVy} = F_{y,RC,F} + F_{y,RC,R} \quad (164)$$

By considering two different roll stiffness for the front and for the rear, it follows:

$$M_{antiroll,F} + M_{antiroll,R} = \left(K_{roll,F} + K_{roll,R}\right)\phi \quad (165)$$

By performing the substitution with *(163)* and by considering roll angles, in first approximation, not significantly larges it follows:

$$\phi = \frac{ma_yH_{roll} + R_{yVy}(h_{HITCH} - RCH)}{K_{roll,F} + K_{roll,R}} \quad (166)$$

By knowing the relationship between the antiroll moment, the roll angle and the roll stiffness expressions of $M_{antiroll,F}$ and $M_{antiroll,R}$ are:

$$M_{antiroll,F} = K_{roll,F}\phi \tag{167}$$

$$M_{antiroll,R} = K_{roll,R}\phi \tag{168}$$

$$M_{antiroll,F} = K_{roll,F}\left(\frac{ma_yH_{roll} + R_{yVy}(h_{HITCH} - RCH)}{K_{roll,F} + K_{roll,R}}\right) \tag{169}$$

$$M_{antiroll,R} = K_{roll,R}\left(\frac{ma_yH_{roll} + R_{yVy}(h_{HITCH} - RCH)}{K_{roll,F} + K_{roll,R}}\right) \tag{170}$$

To determinate the load transfer on the frontal axle a free-body diagram of the towing vehicle front axle is used:

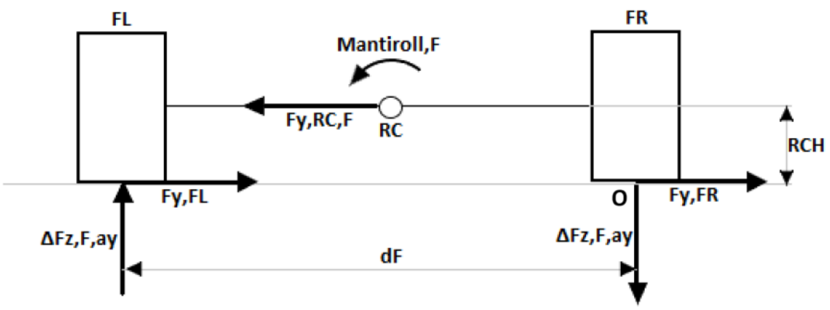


Figure 20. Lateral load transfer - free body diagram front axle towing vehicle.

The lateral force balance of the front axle is:

$$F_{y,RC,F} = F_{y,FL} + F_{y,FR} \tag{171}$$

The balance of the moments in the front axle about the contact point right wheel/road is:

$$\Delta F_{Z,F,ay} = \frac{F_{y,RC,F}RCH + M_{antiroll,F}}{d_F} \tag{172}$$

By performing the opportune substitutions:

$$\Delta F_{Z,F,ay} = \frac{F_{y,RC,F}RCH}{d_F} + \frac{K_{roll,F}\phi}{d_F} = \frac{F_{y,RC,F}RCH}{d_F} + \frac{K_{roll,F}(ma_yH_{roll} + R_{yVy}(h_{HITCH} - RCH))}{d_F(K_{roll,F} + K_{roll,R})} \tag{173}$$

In the same way it is possible to find the lateral load transfer on the rear axle as:

$$\Delta F_{Z,F,ay} = \frac{F_{y,RC,F}RCH}{d_F} + \frac{K_{roll,F}(ma_yH_{roll} + R_{yVy}(h_{HITCH} - RCH))}{d_F(K_{roll,F} + K_{roll,R})} \tag{174}$$

$$\Delta F_{Z,R,ay} = \frac{F_{y,RC,R}RCH}{d_R} + \frac{K_{roll,R}(ma_yH_{roll} + R_{yVy}(h_{HITCH} - RCH))}{d_R(K_{roll,F} + K_{roll,R})} \tag{175}$$

Trailer

Lateral Load transfer considering suspension properties of the trailer can be calculate in the same way of the towing vehicle. The roll stiffness and, consequently, the roll angle are different because we are considering a different kind of suspensions.

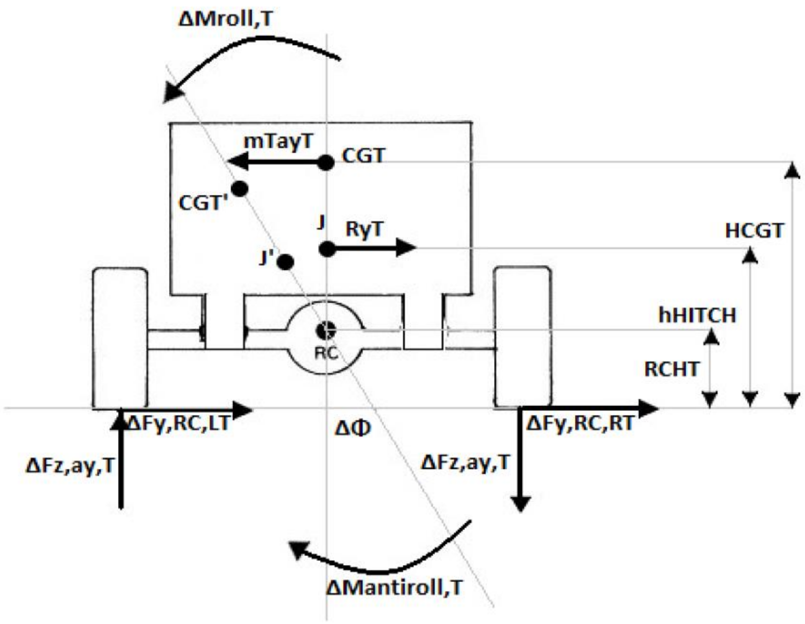


Figure 21. Lateral load transfer - free body diagram trailer rear view.

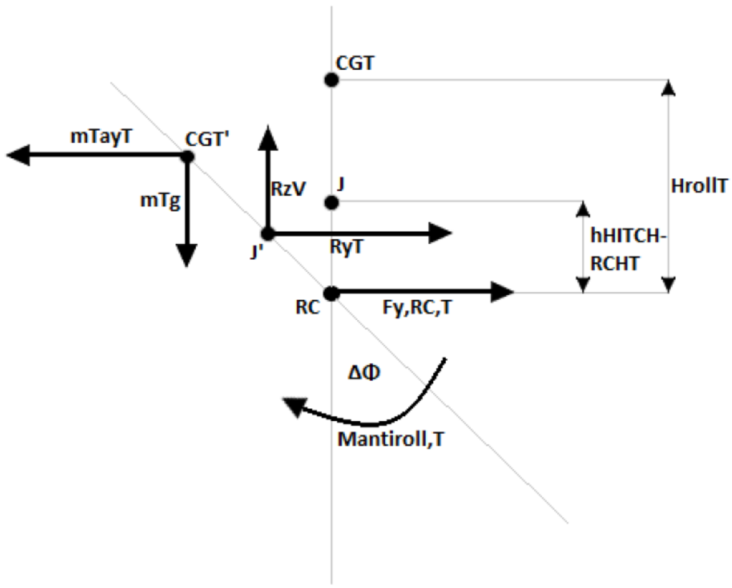


Figure 22. Lateral load transfer – trailer.

By performing the moment balance about RC point it follows:

$$M_{antiroll,T} = m_T g H_{roll,T} \sin(\phi) + m_T a_{yT} H_{roll,T} \cos(\phi) - R_{zV}(h_{HITCH} - RCHT)\sin(\phi) - R_{yT}(h_{HITCH} - RCHT)\cos(\phi) \tag{176}$$

The balance of the lateral forces is:

$$m_T a_{yT} = F_{y,RC,T} + R_{yT} \tag{177}$$

By remembering the expression of $M_{antiroll,T}$:

$$M_{antiroll,T} = K_{roll,T}\phi \tag{178}$$

By performing the substitution with *(176)* and by considering roll angles, in first approximation, not significantly larges it follows:

$$\phi = \frac{m_T a_{yT} H_{roll,T} - R_{yT}(h_{HITCH} - RCHT)}{K_{roll,T}} \tag{179}$$

where R_{yT} is the lateral force exchanged between the trailer and the hitch joint.
The $M_{antiroll,T}$ is:

$$M_{antiroll,T} = m_T a_{yT} H_{roll,T} - R_{yT}(h_{HITCH} - RCHT) \tag{180}$$

After determining the antiroll moment, the lateral load transfer on the trailer axle can be calculated. The free body diagram of the axle of the trailer is:

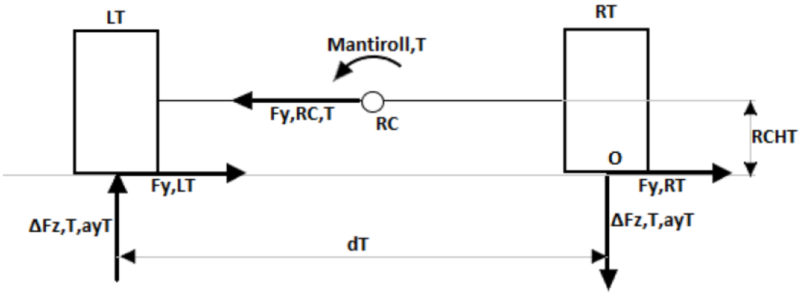


Figure 23. Lateral load transfer - free body diagram trailer axle.

The balance of the lateral forces is:

$$F_{y,RC,T} = F_{y,LT} + F_{y,RT} \tag{181}$$

The moment balance about the contact point O is:

$$\Delta F_{Z,T,ayT} = \frac{F_{y,RC,T}RCHT}{d_T} + \frac{K_{roll,T}\phi}{d_T} \tag{182}$$

By substituting the expression of the roll angle of the trailer (*180*) the total lateral load transfer on the axle of the trailer is:

$$\Delta F_{Z,T,ayT} = \frac{F_{y,RC,T}RCHT}{d_T} + \frac{m_T a_{yT} H_{roll,T} - R_{yT}(h_{HITCH} - RCHT)}{d_T} \tag{183}$$

Thus, the lateral forces on the left and right wheels are:

$$F_{z,T,L} = \left(\frac{F_{y,RC,T}RCHT}{d_T} + \frac{m_T a_{yT} H_{roll,T} - R_{yT}(h_{HITCH} - RCHT)}{d_T}\right) \quad (184)$$

$$F_{z,T,R} = -\left(\frac{F_{y,RC,T}RCHT}{d_T} + \frac{m_T a_{yT} H_{roll,T} - R_{yT}(h_{HITCH} - RCHT)}{d_T}\right) \quad (185)$$

The free body trailer diagram below is used to find $F_{y,RC,T}$ and R_{yT}:

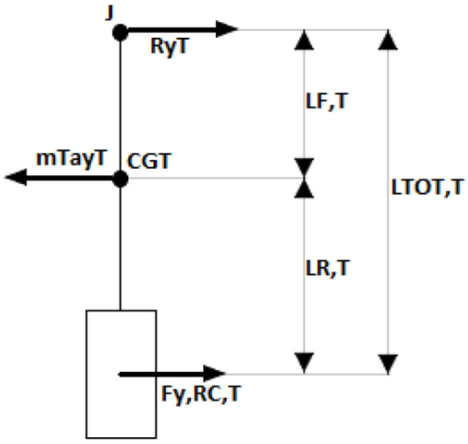


Figure 24. Lateral load transfer - free body diagram trailer top view.

The moment balance about CGT point is:

$$R_{yT}L_{F,T} = F_{y,RC,T}L_{R,T} \quad (186)$$

Moment balance about J point:

$$F_{y,RC,T} = \frac{m_T a_{yT} L_{F,T}}{L_{TOT,T}} \quad (187)$$

Whilst the moment balance about centre of the wheel of the trailer is:

$$R_{yT} = \frac{m_T a_{yT} L_{R,T}}{L_{TOT,T}} \quad (188)$$

The expression of the lateral load transfer on the axle of the trailer is then:

$$\Delta F_{Z,T,ayT} = \frac{m_T a_{yT}}{d_T}\left(\frac{L_{F,T}RCHT}{L_{TOT,T}} + H_{roll,T} - \frac{L_{R,T}(h_{HITCH} - RCHT)}{L_{TOT,T}}\right) \quad (189)$$

The free body of the towing vehicle diagram below is used to find $F_{y,RC,F}$, $F_{y,RC,R}$ and R_{yVy}

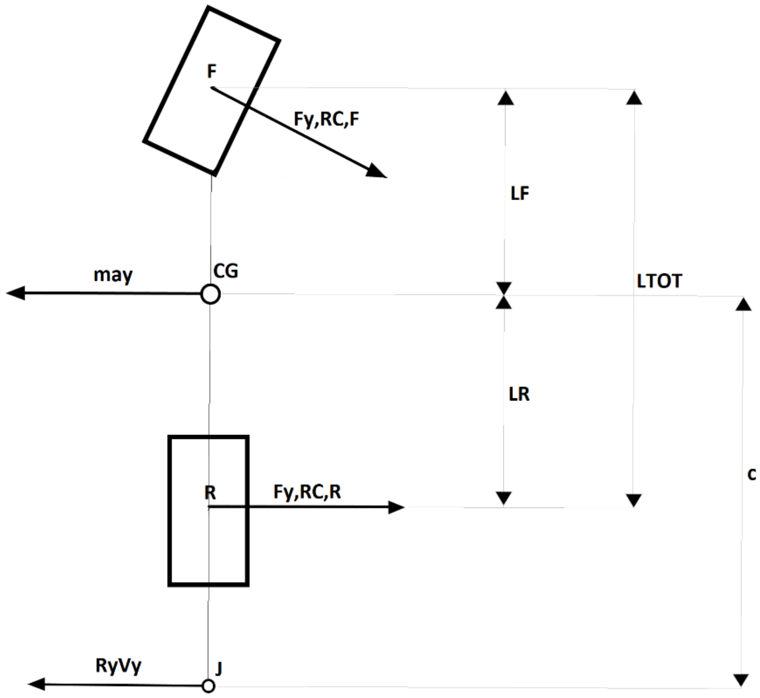


Figure 25. Lateral load transfer - free body diagram towing vehicle top view.

By resolving the hitch joint static reactions distribution, it is possible to obtain the expression of the lateral reaction on the towing vehicle due to the hitch joint $RyVy$:

$$R_{yVy} = (m_T a_{xT} + F_{TW}) \sin(\theta) + \frac{m_T a_{yT} L_{R,T}}{L_{TOT,T}} \cos(\theta) \tag{190}$$

By solving the moment balance of the towing vehicle in R and F the following expressions are obtained:
Balance in R:

$$F_{y,RC,F} = \frac{m a_y L_R}{L_{TOT}} - \frac{R_{yVy}(c - L_R)}{L_{TOT}} \tag{191}$$

Balance in F:

$$F_{y,RC,R} = \frac{m a_y L_F}{L_{TOT}} + \frac{R_{yVy}(c + L_F)}{L_{TOT}} \tag{192}$$

By substituting *(190)* in *(191)*, it follows:

$$F_{y,RC,F} = \frac{m a_y L_R}{L_{TOT}} - \left((m_T a_{xT} + F_{TW}) \sin(\theta) + \frac{m_T a_{yT} L_{R,T}}{L_{TOT,T}} \cos(\theta) \right) \frac{(c - L_R)}{L_{TOT}} \tag{193}$$

By solving with opportune substitutions in (*174*) it is possible to find $\Delta F_{Z,F,ay}$ expression:

$$\Delta F_{Z,F,ay} = \left(\frac{ma_y L_R}{L_{TOT}} - \left((m_T a_{xT} + F_{TW})\sin(\theta) + \frac{m_T a_{yT} L_{R,T}}{L_{TOT,T}}\cos(\theta)\right)\frac{(c - L_R)}{L_{TOT}}\right)\frac{RCH}{d_F} + \frac{K_{roll,F}\left(ma_y H_{roll} + \left((m_T a_{xT} + F_{TW})\sin(\theta) + \frac{m_T a_{yT} L_{R,T}}{L_{TOT,T}}\cos(\theta)\right)(h_{HITCH} - RCH)\right)}{d_F(K_{roll,F} + K_{roll,R})} \tag{194}$$

By substituting *(190)* in *(192)*, it follows:

$$F_{y,RC,R} = \frac{ma_y L_F}{L_{TOT}} + \left((m_T a_{xT} + F_{TW})\sin(\theta) + \frac{m_T a_{yT} L_{R,T}}{L_{TOT,T}}\cos(\theta)\right)\frac{(c + L_F)}{L_{TOT}} \tag{195}$$

In the same way for the rear axle of the towing vehicle by substituting in (*175*) it is possible to find $\Delta F_{Z,R,ay}$ expression:

$$\Delta F_{Z,R,ay} = \left(\frac{ma_y L_F}{L_{TOT}} + \left((m_T a_{xT} + F_{TW})\sin(\theta) + \frac{m_T a_{yT} L_{R,T}}{L_{TOT,T}}\cos(\theta)\right)\frac{(c + L_F)}{L_{TOT}}\right)\frac{RCH}{d_R} + \frac{K_{roll,R}\left(ma_y H_{roll} + \left((m_T a_{xT} + F_{TW})\sin(\theta) + \frac{m_T a_{yT} L_{R,T}}{L_{TOT,T}}\cos(\theta)\right)(h_{HITCH} - RCH)\right)}{d_R(K_{roll,F} + K_{roll,R})} \tag{196}$$

Thus, the vertical loads of the articulated vehicle $F_{z,ij,k}$ are defined using the (*137*) in this way:

$$\begin{aligned} F_{z,FL} &= \frac{1}{2}\left(\frac{mgL_R}{L_{TOT}} - \frac{m_T g L_{R,T}(c - L_R)}{L_{TOT}L_{TOT,T}}\right) \\ &- \frac{1}{2}\left(\frac{ma_x HCG}{L_{TOT}} + \frac{(m_T a_{xT} + F_{TW})h_{HITCH}}{L_{TOT}} - \frac{(m_T a_{xT}\,(HCGT - h_{HITCH}) - F_{TW}h_{HITCH})(c - L_R)}{L_{TOT}L_{TOT,T}}\right) \\ &- \left(\frac{ma_y L_R}{L_{TOT}} - \left((m_T a_{xT} + F_{TW})\sin(\theta) + \frac{m_T a_{yT} L_{R,T}}{L_{TOT,T}}\cos(\theta)\right)\frac{(c - L_R)}{L_{TOT}}\right)\frac{RCH}{d_F} \\ &+ \frac{K_{roll,F}\left(ma_y H_{roll} + \left((m_T a_{xT} + F_{TW})\sin(\theta) + \frac{m_T a_{yT} L_{R,T}}{L_{TOT,T}}\cos(\theta)\right)(h_{HITCH} - RCH)\right)}{d_F(K_{roll,F} + K_{roll,R})} \\ &- \frac{1}{2}\rho V^2 S C_x \frac{h}{2L_{TOT}} \end{aligned} \tag{197}$$

$$F_{z,FR} = \frac{1}{2}\left(\frac{mgL_R}{L_{TOT}} - \frac{m_T g L_{R,T}(c - L_R)}{L_{TOT}L_{TOT,T}}\right) - \frac{1}{2}\left(\frac{ma_x HCG}{L_{TOT}} + \frac{(m_T a_{xT} + F_{TW})h_{HITCH}}{L_{TOT}} - \frac{(m_T a_{xT}\ (HCGT - h_{HITCH}) - F_{TW}h_{HITCH})(c - L_R)}{L_{TOT}L_{TOT,T}}\right) + \left(\frac{ma_y L_R}{L_{TOT}} - \left((m_T a_{xT} + F_{TW})\sin(\theta) + \frac{m_T a_{yT} L_{R,T}}{L_{TOT,T}}\cos(\theta)\right)\frac{(c - L_R)}{L_{TOT}}\right)\frac{RCH}{d_F} + \frac{K_{roll,F}\left(ma_y H_{roll} + \left((m_T a_{xT} + F_{TW})\sin(\theta) + \frac{m_T a_{yT} L_{R,T}}{L_{TOT,T}}\cos(\theta)\right)(h_{HITCH} - RCH)\right)}{d_F(K_{roll,F} + K_{roll,R})} - \frac{1}{2}\rho V^2 S C_x \frac{h}{2L_{TOT}} \tag{198}$$

$$F_{z,RL} = \frac{1}{2}\left(\frac{mgL_F}{L_{TOT}} + \frac{m_T g L_{R,T}(c + L_F)}{L_{TOT}L_{TOT,T}}\right) + \frac{1}{2}\left(\frac{ma_x HCG}{L_{TOT}} + \frac{(m_T a_{xT} + F_{TW})h_{HITCH}}{L_{TOT}} - \frac{(m_T a_{xT}\ (HCGT - h_{HITCH}) - F_{TW}h_{HITCH})(c + L_F)}{L_{TOT}L_{TOT,T}}\right) - \left(\frac{ma_y L_F}{L_{TOT}} + \left((m_T a_{xT} + F_{TW})\sin(\theta) + \frac{m_T a_{yT} L_{R,T}}{L_{TOT,T}}\cos(\theta)\right)\frac{(c + L_F)}{L_{TOT}}\right)\frac{RCH}{d_R} + \frac{K_{roll,R}\left(ma_y H_{roll} + \left((m_T a_{xT} + F_{TW})\sin(\theta) + \frac{m_T a_{yT} L_{R,T}}{L_{TOT,T}}\cos(\theta)\right)(h_{HITCH} - RCH)\right)}{d_R(K_{roll,F} + K_{roll,R})} + \frac{1}{2}\rho V^2 S C_x \frac{h}{2L_{TOT}} \tag{199}$$

$$F_{z,RR} = \frac{1}{2}\left(\frac{mgL_F}{L_{TOT}} + \frac{m_T g L_{R,T}(c + L_F)}{L_{TOT}L_{TOT,T}}\right) + \frac{1}{2}\left(\frac{ma_x HCG}{L_{TOT}} + \frac{(m_T a_{xT} + F_{TW})h_{HITCH}}{L_{TOT}} - \frac{(m_T a_{xT}\ (HCGT - h_{HITCH}) - F_{TW}h_{HITCH})(c + L_F)}{L_{TOT}L_{TOT,T}}\right) + \left(\frac{ma_y L_F}{L_{TOT}} + \left((m_T a_{xT} + F_{TW})\sin(\theta) + \frac{m_T a_{yT} L_{R,T}}{L_{TOT,T}}\cos(\theta)\right)\frac{(c + L_F)}{L_{TOT}}\right)\frac{RCH}{d_R} + \frac{K_{roll,R}\left(ma_y H_{roll} + \left((m_T a_{xT} + F_{TW})\sin(\theta) + \frac{m_T a_{yT} L_{R,T}}{L_{TOT,T}}\cos(\theta)\right)(h_{HITCH} - RCH)\right)}{d_R(K_{roll,F} + K_{roll,R})} + \frac{1}{2}\rho V^2 S C_x \frac{h}{2L_{TOT}} \tag{200}$$

$$F_{z,L,T} = \frac{m_T g L_{F,T}}{2L_{TOT}} + \frac{1}{2}\left(\frac{m_T a_{xT}(HCGT - h_{HITCH}) - F_{TW}h_{HITCH}}{L_{TOT,T}}\right) - \frac{m_T a_{yT}}{d_T}\left(\frac{L_{F,T}RCHT}{L_{TOT,T}} + H_{roll,T} - \frac{L_{R,T}}{L_{TOT,T}}(h_{HITCH} - RCHT)\right) \tag{201}$$

$$F_{z,R,T} = \frac{m_T g L_{F,T}}{2L_{TOT}} + \frac{1}{2}\left(\frac{m_T a_{xT}(HCGT - h_{HITCH}) - F_{TW}h_{HITCH}}{L_{TOT,T}}\right) + \frac{m_T a_{yT}}{d_T}\left(\frac{L_{F,T}RCHT}{L_{TOT,T}} + H_{roll,T} - \frac{L_{R,T}}{L_{TOT,T}}(h_{HITCH} - RCHT)\right) \tag{202}$$

3.2 Nonlinear model predictive control - optimal control problem

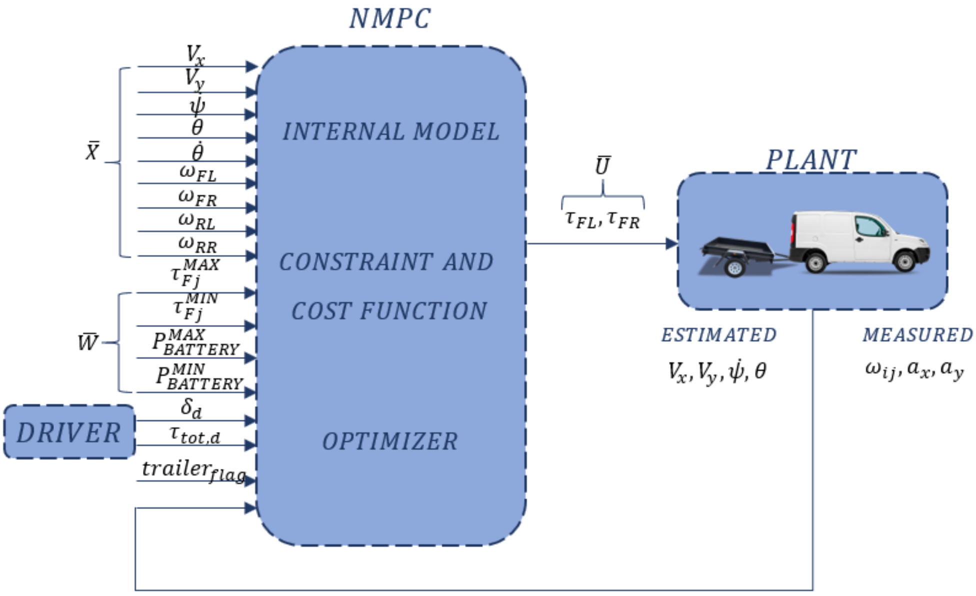


Figure 26. Schematic diagram for NMPC and plant – articulated vehicle configuration.

Similar to the rigid vehicle configuration, an articulated vehicle configuration in state-space form is developed by combining the articulated vehicle dynamics equations *(130)*,*(131)*,*(132)*,*(133)* and the wheel dynamics equations *(134)* and the full articulated vehicle configuration in standard state-space form is expressed as *(49)* where X is the state vector defined as follows:

$$X = [V_x, V_y, \theta, \dot{\theta}, \dot{\psi}, \omega_{FL}, \omega_{FR}, \omega_{RL}, \omega_{RR}]^T \tag{203}$$

where V_x is the vehicle longitudinal velocity at its CM, V_y is the vehicle lateral velocity at its CM, θ is the hitch angle, $\dot{\theta}$ is the hitch rate, $\dot{\psi}$ is the yaw rate of the vehicle and $\omega_{FL}, \omega_{FR}, \omega_{RL}, \omega_{RR}$ are the wheel velocity of the vehicle. Then, W is the parameters vector defined as in *(51)* and U is the controlled input vector defined as in *(52)*.The discrete-time state-space form of the articulated vehicle model written as in *(53)*.

Some assumptions must be done before developing the NMPC controller: the driver inputs such as steer angles, torques on wheels, vehicle speed and the load transfer are constant during prediction horizon N which is a common approach in developing the NMPC for vehicle stability control.

Therefore, to determine the control input that should be applied to the plant at the next time instant a constrained quadratic optimization problem has to be solved by using the cost function as in *(54)*.

3.3 Hitch angle controller approaches

3.3.1 First approach

This approach includes the hitch angle error in the cost function which is taken in account only if the hitch angle actual value overcomes a pre-determined thresholds thus, the controller acts only if there is an important instability of the articulated vehicle.

From Figure 27 it is possible to see the shape of the function with the respect to the actual hitch angle error computed by considering different $\Delta\theta_{th}$ that are the thresholds values beyond which the function is different from zero.

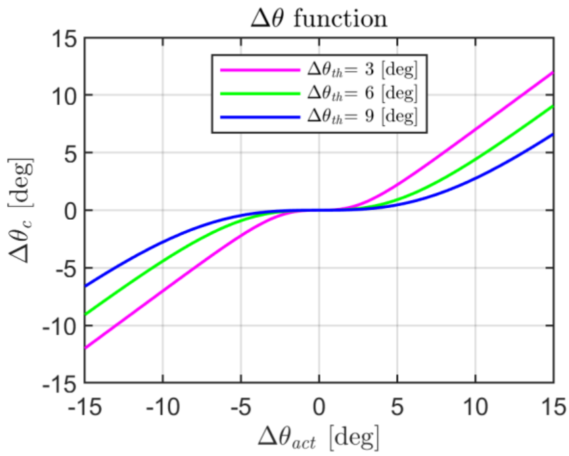


Figure 27. $\boldsymbol{\Delta\theta_c}$ computed with different $\boldsymbol{\Delta\theta_{th}}$

Where $\Delta\theta_c$ is the novel function used inside the cost function which changes based on the hitch angle error $\Delta\theta_{act}$. In this approach the constraints are the same as for the rigid vehicle configuration *(60)*, *(61), (62)*, *(63)* and *(64)*. The articulated vehicle configuration, as internal model, is considered.

3.3.2 Second approach

The second approach considers a soft constraint on the hitch angle error and a slack variable s_θ is added in the cost function, which is linked to the constraint on the hitch angle inside ACADO. The main aim is to activate the controller only when the thresholds are overcome. As the previous approach the articulated vehicle configuration is used as internal model. In this approach the soft constraint on the hitch angle, with respect to the rigid vehicle configuration constraints *(60)*, *(61), (62)*, *(63)* and *(64)*, is added. Thus, the hitch angle is limited between two thresholds as function of the slack variable.

3.3.3 Third approach

This approach is based on the modification of the yaw rate error formulation, by substituting it with a weighted linear combination of the yaw rate error and the hitch angle error where the latter has an influence only when it exceeds pre-determined thresholds, as shown in [8].
The controller blends the yaw rate and hitch angle contributions only when the trailer dynamics are deemed critical. More specifically, three different cases are considered: in the first case the controller tracks only the reference yaw rate of the car, in the second case the controller blends the contribution of yaw rate error and hitch angle error and in the third case, during extreme oscillation of the trailer, the controller is almost completely focused on the hitch angle control but, on the other hand, a little influence to control the vehicle trajectory is left to the driver or to the automated driving controller. The articulated vehicle configuration is used as internal model as in the previous approaches. Also with this approach the constraints are the same as for the rigid vehicle configuration *(60)*, *(61), (62)*, *(63)* and *(64)*.

3.3.4 Fourth approach

This approach is based on the modification of the reference yaw rate formulation similar to the previous one, by substituting it with a weighted linear combination of the yaw rate and the hitch angle error where the latter has an influence only when it exceeds pre-determined thresholds. In this case the yaw rate inside the linear combination is the desired yaw rate for the rigid vehicle configuration defined using a look-up table. The internal model is the rigid vehicle configuration and the controller blends the contributions of the yaw rate and the hitch angle error only when the trailer dynamics are deemed critical. As in the previous approaches the constraints are the same as for the rigid vehicle configuration *(60)*, *(61), (62)*, *(63)* and *(64)*.

3.3.5 Selection of results

3.3.5.1 Controllers tuning routine

For a fair comparison of the proposed hitch angle controllers, the following key performance indicators were used:

- The root-mean-square error of the yaw rate error:

$$RMSE_{\Delta\dot{\psi}} = \frac{\sqrt{\frac{1}{t_f - t_i}\int_{t_i}^{t_f}\left(\dot{\psi}_d(t) - \dot{\psi}(t)\right)^2 dt}}{M_{\Delta\dot{\psi}}} \tag{204}$$

where t_i is 1 s and t_f represents the end of the manoeuvre, $\dot{\psi}_d$ is the desired yaw rate of the rigid vehicle, $\dot{\psi}$ is the actual yaw rate and $M_{\Delta\dot{\psi}}$ is a normalisation factor expressed as the maximum expected value of $RMSE_{\Delta\dot{\psi}}$.

- The root-mean-square error of the hitch angle error:

$$RMSE_{\Delta\theta^*} = \frac{\sqrt{\frac{1}{t_f - t_i}\int_{t_i}^{t_f}(\Delta\theta^*)^2 dt}}{M_{\Delta\theta^*}} \tag{205}$$

$$\Delta\theta^* = \begin{cases} |\theta_{des}(t) - \theta(t)| - \Delta\theta_{bound} & if \quad |\theta_{des}(t) - \theta(t)| > \Delta\theta_{bound} \\ 0 & if \quad |\theta_{des}(t) - \theta(t)| \leq \Delta\theta_{bound} \end{cases} \tag{206}$$

where $M_{\Delta\theta^*}$ is a normalisation factor expressed as the maximum expected value of $RMSE_{\Delta\theta^*}$ and $\Delta\theta_{bound}$ is the limit value from which the $RMSE_{\Delta\theta^*}$ is calculated.

- The integral of the absolute value of the control action, $IACA$, which evaluates the control effort:

$$IACA = \frac{\frac{1}{t_f - t_i}\int_{t_i}^{t_f}|\tau_{FL}(t) - \tau_{FR}(t)|dt}{M_{IACA}} \tag{207}$$

where M_{IACA} is a normalisation factor expressed as the maximum expected value of $IACA$.

- The maximum rear sideslip angle:

$$\alpha_R^{max} = \frac{\max|\alpha_R|}{M_{\alpha_R^{max}}} \tag{208}$$

where $M_{\alpha_R^{max}}$ is a normalisation factor expressed as the maximum value of α_R^{max}.

- The maximum hitch angle:

$$\theta^{max} = \frac{\max|\theta|}{M_{\theta^{max}}} \tag{209}$$

where $M_{\theta^{max}}$ is a normalisation factor expressed as the maximum value of θ^{max}.

Thus, a cost function J_{KPI}, which combines in a weighted sum all the previous performance indicators is used:

$$J_{KPI} = W_1 RMSE_{\Delta\dot{\psi}} + W_2 RMSE_{\Delta\theta^*} + W_3 IACA + W_4 \alpha_R^{max} + W_5 \theta^{max} \quad (210)$$

where W_{1-5} are the weights for the individual performance indicator in the cost function J_{KPI}, and each term is non-dimensional.

The optimization problem is described by:

$$J_{KPI}^{*} = arg_{P_{opt}} \min J_{KPI}\big|_{t_i}^{t_f} \quad (211)$$

$$s.t. P_{LB} \leq P_{opt} \leq P_{UB} \quad (212)$$

where J_{KPI} is the cost function described in (210), J_{KPI}^{*} is the optimal value of the cost function; P_{LB} and P_{UB} are the lower and upper bounds on P_{opt}; and t_i and t_f are the initial and the final times of the test. In order to minimize the cost function, that is indicated above in (210), a design of experiment simulation campaign was performed to select the optimal value of the tuning parameters for each controllers. This simulation campaign was carried out under some constraints that are specified in (212).

3.3.5.2 Manoeuvre

- Sweep steering test with a sinusoidal steering wheel input at a progressively increasing frequency and 50 deg amplitude, starting at V = 70 km/h frequency from 0 to 0.25 Hz.

3.3.5.3 First approach

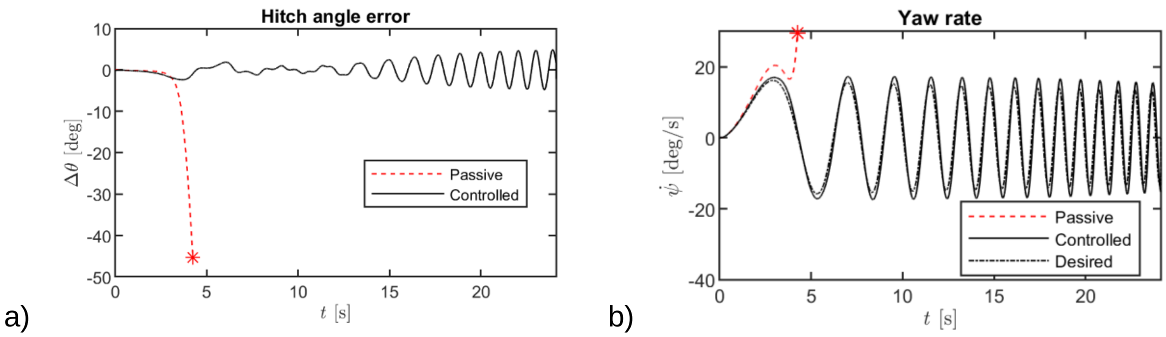
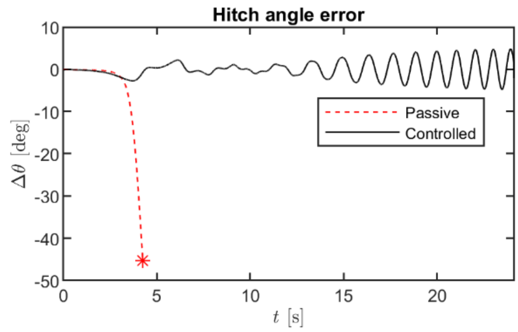


Figure 28. Hitch angle error (a) and yaw rate (b).

Figure 28 shows the time histories of hitch angle error and yaw rate simulated during sweep steering test which significantly excite the trailer dynamics. The passive configuration reaches a maximum hitch angle error of ~45 deg, after which the simulation is interrupted. With this approach the controller aggressively intervenes only when a threshold value is exceeded. It is interesting to see the important damping effect on the hitch angle error oscillation, due to the controller intervention. More specifically, the TV system dampens the hitch error oscillation which is kept bounded to a low amplitude of ~ 8 deg. Moreover, a good performance in terms of yaw rate tracking is achieved.

3.3.5.4 Second approach

a) 

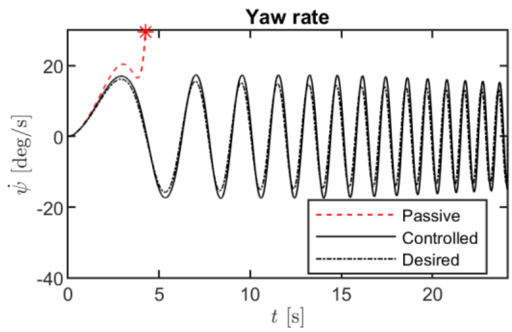
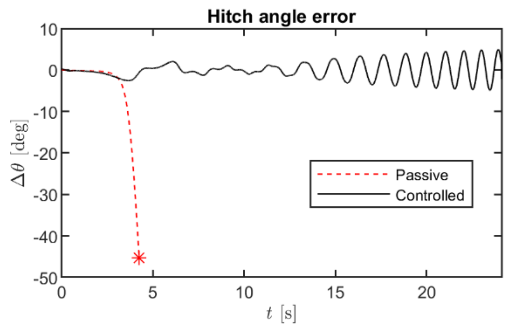
b) 

Figure 29. Hitch angle error (a) and yaw rate (b).

In Figure 29, it is clearly visible that the results of the second approach are similar to the first one. The overall articulated vehicle is operating in less critical condition with respect to the passive vehicle. More in detail, the hitch angle error is bounded between ~ - 8 deg and ~ 8 deg and the controlled yaw rate follows quite well the desired.

3.3.5.5 Third approach

a) 

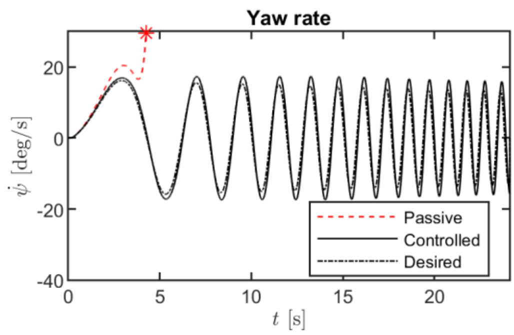
b) 

Figure 30. Hitch angle error (a) and yaw rate (b).

In Figure 30 the time histories of the hitch angle error and yaw rate, of the third approach, are reported. Also in this case it is shown a reduced hitch angle error and a good yaw rate tracking performance. More specifically, the hitch angle error is kept bounded between ~ - 8 deg and ~ 8 deg.

3.3.5.6 Fourth approach

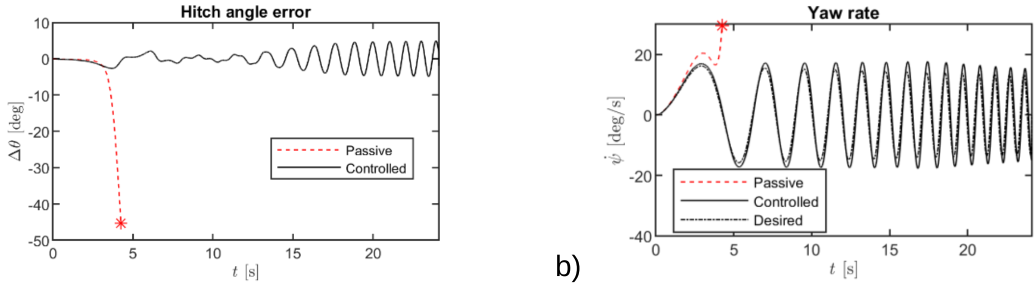


Figure 31. Hitch angle error (a) and yaw rate (b).

Figure 31 shows the behaviour of the fourth controller approach. Also in this case, an overall damping effect of the hitch angle error oscillation is displayed. With this approach a slightly higher hitch angle error value is shown. This difference can be attributed to the use of the rigid vehicle configuration as internal model.

To assess the controllers behaviour during the manoeuvre, the key performance indicators reported in Table 1 are used:

Table 1.: Key performance indicators

		Passive*	1st approach	2nd approach	3rd approach	4th approach[1]
$RMSE_{\Delta\dot{\psi}}$	[deg/s]	5.97	3.06	3.15	3.07	3.29
$RMSE_{\Delta\theta^*}$	[deg]	8.22	0.00	0.00	0.00	0.00
$IACA$	[Nm]	-	308	289	331	312
$\lvert\alpha_R^{max}\rvert$	[deg]	6.07	2.15	2.16	2.13	2.14
$\lvert\theta^{max}\rvert$	[deg]	45.00	5.73	6.17	6.17	6.41
J_{KPI}^*	[-]	/	1.14	1.15	1.21	1.21

[1]: rigid vehicle used as internal model

*: hitch angle reaches a threshold value. In this case the simulation is aborted early

-: non-calculable value

/: simulation interrupted; value not calculated

To evaluate the performances of the different controller formulations, both values of the cost function defined in Eq. (211) and key performance indicators are considered. As it is clearly visible from Table 1 the first approach provides the best response because its key performance indicators are lower than the other controller formulations, e.g. $\lvert\theta^{max}\rvert$ = 5.73 deg and J_{KPI}^* = 1.14. The second approach achieves good results in terms of maximum hitch angle, $\lvert\theta^{max}\rvert$ = 6.17 deg, and in terms of control effort requested to restrain trailer oscillation $IACA$ = 289 Nm ; however, its $RMSE_{\Delta\dot{\psi}}$ value, which indicates the yaw rate tracking performance during the manoeuvre, is quite high. The third approach, in terms of performances, is ranked after the previous two formulations, showing the same value of maximum hitch angle of the second approach but a higher value of the cost function J_{KPI}^* = 1.21. Lastly the fourth approach seems to be the less performant hitch angle controller because it shows a higher value of maximum hitch angle, $|\theta^{max}|$ = 6.41 deg. This can be attributed, as said before, to the use of the rigid vehicle configuration as internal model. Anyway, all the proposed controllers show a null value of $RMSE_{\Delta\theta^*}$, which means that the hitch angle error is lower than $\Delta\theta_{bound}$ as defined in Eq. (206).

The simulation analysis, based on the vehicle simulation model, shows that: i) the vehicle dynamics performance in emergency conditions is consistently enhanced by the four proposed NMPC formulations and ii) the NMPC formulations that directly constrain the hitch angle error, or carry out continuous hitch angle tracking, outperform the formulations that modify the reference yaw rate or the yaw rate error to compensate the hitch angle oscillations.

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